Section 11.1 *Sequences*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know what is meant convergence of a sequence
2. learn some tools for finding limit of a sequence
3. what is meant by monotone sequences
4. learn techniques for finding whether a sequence is monotone or not.

Sequence?

Convergence of a sequence?

What is a sequence?

- A list of successive terms written as $a_1, a_2, \cdots\cdots, a_n, \cdots\cdots$
- If we can find general term then also written as $\{a_n\}_{n=1}^{\infty}$
    - Also written as a function $f(n) = a_n, \quad n = 1, 2, 3, \cdots$

See examples done in class

Main Question

Knowing convergence of a sequence

OR

finding limit of a sequence

- A sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit L if given any $\varepsilon > 0$, there is a positive integer N such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

  Notation: $\lim_{n\to\infty} a_n = L$
- Otherwise the sequence diverges.

Means: As 'n' increases the terms of sequence stay close to the value 'L'

- **i.e. for** $n \geq N$ **all** a_n **lie within** $\varepsilon -$ band **aroumd** L

- See graphical explanation (from WebCT)
- And other details in class

Computation of limit of a sequence

Main tool:

Find $\lim_{n\to\infty} a_n$

if it exists and is a finite number 'L' then

- sequence converges
- and $\lim_{n\to\infty} a_n = L$

if it does not exist (or is $\pm\infty$) then

- sequence diverges

See examples 1, 2, 3 done in class

Other important tricks/tools or results:

1. Use of l'Hopital's rule

See example 4 in class

2. Look at behavior of even & odd terms separately

- Particularly useful for limits of the type $(-1)^n a_n$ or $(-1)^{n+1} a_n$
- See example 5 in class

A sequence converges to a limit L

$\Leftrightarrow$

sequences of even-numbered terms & odd numbered terms both converge to L

Some results useful in computations of limit of a sequence

Theorem 1

Suppose that the sequences $\{a_n\}$ *and* $\{b_n\}$ *converge to limits* L_1 *and* L_2 *respectively, and* c *is a constant. Then*

1. $\lim\limits_{n\to\infty} c = c$
2. $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = cL_1$
3. $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L_1 \pm L_2$
4. $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n = L_1 L_2$
5. $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \left(\dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}\right) = \dfrac{L_1}{L_2}$ *(if* $L_2 \neq 0$*)*

Theorem 2 [Sandwich theorem]

Let $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be sequences such that*

$$a_n \le b_n \le c_n$$

[for all values of n *greater than some number* N*].*

If $\{a_n\}$ *and* $\{c_n\}$ *converge to 'L' then* $\{b_n\}$ *also converges to 'L'.*

What is a monotone sequence?

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

- strictly increasing if $a_1 < a_2 < a_3 < \cdots\cdots < a_n < \cdots$
- increasing if $a_1 \le a_2 \le a_3 \le \cdots\cdots \le a_n \le \cdots$
- strictly decreasing if $a_1 > a_2 > a_3 > \cdots\cdots > a_n > \cdots$
- decreasing if $a_1 \ge a_2 \ge a_3 \ge \cdots\cdots \ge a_n \ge \cdots$

See examples done in class

A (strictly) increasing or (strictly) decreasing sequence is called a (strictly) monotone sequence

Techniques for checking whether a sequence is monotone or not

- Given a sequence $\{a_n\}_{n=1}^{\infty}$.

T1: Difference of successive terms

Look at $a_{n+1} - a_n$

See example 6 in class

- $a_{n+1} - a_n > 0$: strictly increasing
- $a_{n+1} - a_n \ge 0$: increasing
- $a_{n+1} - a_n < 0$: strictly decreasing
- $a_{n+1} - a_n \le 0$: decreasing

T2: Ratio of successive terms

{only for positive term sequences}

Look at $\frac{a_{n+1}}{a_n}$

See example 7 in class

- $\frac{a_{n+1}}{a_n} > 1$: strictly increasing
- $\frac{a_{n+1}}{a_n} \ge 1$: increasing
- $\frac{a_{n+1}}{a_n} < 1$: strictly decreasing
- $\frac{a_{n+1}}{a_n} \le 1$: decreasing

T3: Using differentiation

- Use $f(n)=a_n$ to write $f(x)$ for $x \geq 1$.

Now look at $f'(x)$ to know the increasing or decreasing properties of $f(x)$ and hence, in particular, of $f(n)=a_n$.

- $f'(x) > 0$: strictly increasing
- $f'(x) \geq 0$: increasing
- $f'(x) < 0$: strictly decreasing
- $f'(x) \leq 0$: decreasing

See example 8 in class

End of 11.1