A REMARK ON HARMONIC MAPS TO A SURFACE

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Abstract. P. Baird in [2] studied harmonic maps from closed Riemannian 3-manifolds to a surface and determined necessary conditions under which these maps become harmonic morphisms. These necessary conditions are generalized for higher dimensional domain manifolds. Furthermore, we discuss the case when these conditions are sufficient as well as the cases when these are restricted.

1. Introduction

Harmonic morphisms are maps between Riemannian manifolds which preserve germs of harmonic functions i.e. these (locally) pull back real-valued harmonic functions to real-valued harmonic functions. These are characterized as harmonic maps which are horizontally (weakly) conformal. Hence, harmonic morphisms can be viewed as a subclass of harmonic maps. Although harmonic mappings are a much wider class, a natural question originating from the above fact is to determine conditions under which a harmonic map reduces to a harmonic morphism. A step in this direction was taken by P. Baird in [2], where he studied harmonic maps from compact 3-dimensional manifolds to a surface and determined conditions on the domain manifold under which these maps become harmonic morphisms.

The aim of this note is to extend this problem to higher dimensional domains and find necessary conditions when a harmonic mapping from Riemannian manifolds of dimension $m \geq 3$ to a surface must be a harmonic morphism. The sufficiency and restrictions on these conditions are also discussed.

2. Harmonic maps and harmonic morphisms

Let $\phi : M \to N$ be a smooth map. Then the tension field $\tau(\phi)$ of ϕ, as defined by Eells-Sampson in [5], is the vector field given by $\mathbf{trace}\tilde{\nabla} d\phi$.

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Definition 2.1. A smooth map $\phi : M \to N$ is said to be harmonic if and only if it extremizes the associated energy integral $E(\phi) = \int_\Omega e(\phi) dv^M$ for every compact domain $\Omega \subset M$ where $e(\phi) = \frac{1}{2}\|d\phi\|^2$ is the energy density of ϕ.

It is well-known, cf. [5, 3, 4], that a map is harmonic if and only if its tension field is zero.

The notions of horizontally conformal maps and harmonic morphisms were formally introduced independently by B. Fuglede [6] and T. Ishihara [8]. In a sense, the former can be thought of as a generalization of the concept of Riemannian submersions and later can be thought of as a special class of harmonic maps. Here we present the basic definitions, and refer to [1, 6, 10] for the fundamental results and properties.

For a smooth map $\phi : M^m \to N^n$, let $C_\phi = \{x \in M \mid rank d\phi_x < n\}$ and let M^* denote the set $M \setminus C_\phi$. For each $x \in M^*$, the *vertical* and *horizontal* spaces are defined by $T_x^V M = \mathrm{Ker} d\phi$ and $T_x^H M = (\mathrm{Ker} d\phi)^\perp$ respectively.

Definition 2.2. A smooth map $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, $d\phi_x : T_x^H M \to T_{\phi(x)} N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\langle d\phi(X), d\phi(Y)\rangle^N = \lambda^2 \langle X, Y\rangle^M \quad \forall X, Y \in T^H M.$$

By setting $\lambda = 0$ on C_ϕ, we can extend $\lambda : M \to \mathbb{R}_0^+$ as a continuous function on M such that λ^2 is a smooth function on M, in fact $\lambda^2 = \frac{\|d\phi\|^2}{n}$. The function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map ϕ.

Harmonic morphisms are maps between Riemannian manifolds which preserve Laplace's equation in the following sense.

Definition 2.3. A smooth map $\phi\colon M^m \to N^n$ is called a *harmonic morphism* if, for every real-valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ non-empty, $f \circ \phi$ is a real-valued harmonic function on $\phi^{-1}(V) \subset M$.

These are related to harmonic maps and horizontally (weakly) conformal maps via the following characterization, obtained in [6, 8].

A smooth map ϕ is a harmonic morphism if and only if it is harmonic and horizontally *(weakly) conformal.*

Due to the above characterization, harmonic morphisms may be viewed as a subclass of harmonic maps. However, it is important to notice that in certain cases harmonic morphisms have properties which are exactly dual to the properties of harmonic maps (see explanation by J. C. Wood in [11]).

3. Formulation of the problem

Let $m \geq 1$. Let $\phi\colon (M^{m+2}, \langle\cdot,\cdot\rangle^M) \to (N^2, h)$ be a non-constant submersive map. For each $x \in M$, the first fundamental form $\phi^* h$ of ϕ can be thought, via the identification maps $\sharp$ and $\tilde{\phi^*}h$, as a map $\Phi = \sharp \circ \tilde{\phi^*}h : T_xM \to T_xM$ where $\tilde{\phi^*}h : T_xM \to T_x^*M$ is defined as $\left(\tilde{\phi^*}h(X)\right)(Y) = \phi^*h(X,Y)$ and $\sharp : T_x^*M \to T_xM$ is defined as $\langle \alpha^\sharp, Y\rangle^M = \alpha(Y)$, $\alpha \in T_x^*M$. For each $x \in M$ we call the eigenvalues of Φ the eigenvalues of ϕ^*h. Since ϕ is submersive, ϕ^*h has two non-trivial positive eigenvalues at each point.

Lemma 3.1. *Let $m \geq 1$. Let $\phi\colon (M^{m+2}, \langle\cdot,\cdot\rangle^M) \to (N^2, h)$ be a non-constant submersive map.*

(1) *If λ_k, $k = 1, 2$, is an eigenvalue of ϕ^*h and X_k is the corresponding eigenvector then $\lambda_k \langle X_k, \cdot\rangle^M = \phi^*h(X_k, \cdot)$.*
(2) *Let $T_x^H M = (\ker d\phi_x)^\perp$ denote the horizontal space at $x \in M$, with an orthonormal basis (X_1, X_2) of eigenvectors of distinct (non-trivial) eigenvalues λ_1, λ_2 of ϕ^*h then $d\phi \cdot X_1$, $d\phi \cdot X_2$ are orthogonal and $\|d\phi \cdot X_1\|^2 = \lambda_1$, $\|d\phi \cdot X_2\|^2 = \lambda_2$.*

In the subsequent sections, we assume $\phi\colon (M^{m+2}\langle\cdot,\cdot\rangle^M) \to (N^2\langle\cdot,\cdot\rangle^N)$ to be a submersive harmonic map and denote by λ_1, λ_2 be the non-trivial eigenvalues of the first fundamental form of ϕ, with the corresponding eigenvectors X_1, X_2. Let $(X_1, X_2, U_i)_{i=1}^m$ be an orthonormal basis of $T_xM = T_x^HM \oplus T_x^VM$ such that (X_1, X_2) is an orthonormal basis of T_x^HM and $(U_i)_{i=1}^m$ is an orthonormal basis of T_x^VM. We consider the function

$$\mu = \lambda_1 - \lambda_2 \tag{3.1}$$

on M^{m+2}. Clearly $\mu \equiv 0$ if and only if ϕ is a harmonic morphism.

The purpose of the calculations, to follow, is to obtain an integral formula and determine conditions which force μ to be zero. Therefore, in the following calculations we will assume that at all points of M^{m+2}, $\mu \neq 0$ i.e. $\lambda_1 \neq \lambda_2$.

Section 4 is devoted to the computation of the Laplacian $\Delta\mu$. The integral formula and its consequences are presented in Section 5.

4. Computation of the Laplacian $\Delta\mu$

The Laplacian of μ is given by:

Proposition 4.1 (Laplacian)**.**

$$\begin{aligned}
\frac{1}{2}\Delta\mu &= (\mu K_H + \lambda_1 K_1 - \lambda_2 K_2) + \frac{1}{2}\sum_{i=1}^{m}\left\{\frac{(U_i(\lambda_1))^2}{\lambda_1} - \frac{(U_i(\lambda_2))^2}{\lambda_2} - d\mu\left(\nabla^M_{U_i}U_i\right)\right\}\\
&+ 2\mu\left\{(\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2 + (\langle\nabla^M_{X_2}X_2, X_1\rangle^M)^2\right\} + 2\mu\sum_{i=1}^{m}(\langle\nabla^M_{U_i}X_1, X_2\rangle^M)^2\\
&+ \sum_{i=1}^{m}\left\{X_1(\lambda_1)\cdot\langle\nabla^M_{U_i}U_i, X_1\rangle^M - X_2(\lambda_2)\cdot\langle\nabla^M_{U_i}U_i, X_2\rangle^M\right\}\\
&\qquad + \sum_{i=1}^{m}\left\{\mu\langle\nabla^M_{X_1}X_1 + \nabla^M_{X_2}X_2, \nabla^M_{U_i}U_i\rangle^M\right\}\\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\lambda_1(\langle\nabla^M_{U_i}U_j, X_1\rangle^M)^2 - \lambda_2(\langle\nabla^M_{U_i}U_j, X_2\rangle^M)^2\right\}\\
&\qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\langle\nabla^M_{U_i}U_i, U_j\rangle^M\left\{-\frac{\lambda_1}{\lambda_2}U_j(\lambda_2) + \frac{\lambda_2}{\lambda_1}U_j(\lambda_1)\right\}
\end{aligned}$$

where K_l, $l = 1, 2$ and K_H are the curvatures defined by

$$K_l = \sum_{i=1}^{m}\langle R^M(X_l, U_i)X_l, U_i\rangle^M \qquad \text{and} \qquad K_H = \langle R^M(X_1, X_2)X_1, X_2\rangle^M.$$

Proof. We can write

$$\begin{aligned}
-\frac{1}{2}\Delta\mu &= -\frac{1}{2}\left\{\sum_{i=1}^{m}\nabla d\mu(U_i, U_i) + \sum_{k=1}^{2}\nabla d\mu(X_k, X_k)\right\}\\
&= -\frac{1}{2}\left\{\Delta^v\mu + \Delta^h\mu\right\}.
\end{aligned}$$

We call $\Delta^v\mu$ as the *vertical Laplacian* and $\Delta^h\mu$ as the *horizontal Laplacian.* Note that these are both well-defined (i.e. independent of the bases U_i and X_k chosen). The required result follows from the expressions of these Laplacians calculated in Section 4.1 and Section 4.2 respectively. □

4.1. **The vertical Laplacian.** By its definition, the vertical Laplacian is given by

$$-\Delta^v \mu = \sum_{i=1}^{m} \left\{ U_i(U_i(\mu)) - d\mu\left(\nabla^M_{U_i} U_i\right) \right\}.$$

Prior to computing an expression for the vertical Laplacian we prove a few Lemmas needed to simplify our computations and calculate $U_i(U_i(\mu))$ in terms of sectional curvatures.

Lemma 4.2. *For each $i = 1, \dots, m$, $k = 1, 2$,*

(1) $d\phi[X_k, U_i] = -\nabla^{\phi^{-1}TN}_{U_i} d\phi{\cdot}X_k$.
(2) $\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi{\cdot}X_k, d\phi{\cdot}X_k \rangle^N = \frac{U_i(\lambda_k)}{2}$.
(3)

$$\begin{aligned}\sum_{i=1}^{m} \langle \nabla^M_{X_1} X_1, X_2 \rangle^M \langle \nabla^M_{U_i} U_i, X_2 \rangle^M &= \sum_{i=1}^{m} \langle \nabla^M_{X_1} X_1, \nabla^M_{U_i} U_i \rangle^M \\ &- \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \langle \nabla^M_{X_1} X_1, U_j \rangle^M \langle \nabla^M_{U_i} U_i, U_j \rangle^M.\end{aligned}$$

(4)

$$\begin{aligned}\sum_{i=1}^{m} \langle \nabla^M_{X_2} X_2, X_1 \rangle^M \langle \nabla^M_{U_i} U_i, X_1 \rangle^M &= \sum_{i=1}^{m} \langle \nabla^M_{X_2} X_2, \nabla^M_{U_i} U_i \rangle^M \\ &- \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \langle \nabla^M_{X_2} X_2, U_j \rangle^M \langle \nabla^M_{U_i} U_i, U_j \rangle^M.\end{aligned}$$

(5) $\langle \nabla^M_{U_i} U_j, X_k \rangle^M = \langle \nabla^M_{U_j} U_i, X_k \rangle^M$.

Proof. (1) The proof follows from $d\phi[X_k, U_i] = \nabla^{\phi^{-1}TN}_{X_k} d\phi{\cdot}U_i - \nabla^{\phi^{-1}TN}_{U_i} d\phi{\cdot}X_k$ and the fact that $d\phi{\cdot}U_i = 0$.

(2) Differentiating $\phi^*\langle X_k, X_k\rangle^N = \lambda_k$ gives $2\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi{\cdot}X_k, d\phi{\cdot}X_k \rangle^N = U_i(\lambda_k)$.

(3) Writing $\nabla^M_{X_1} X_1 = \langle \nabla^M_{X_1} X_1, X_2 \rangle^M X_2 + \sum_{i=1}^{m} \langle \nabla^M_{X_1} X_1, U_i \rangle^M U_i$, we have

$$\begin{aligned}\sum_{i=1}^{m} \langle \nabla^M_{X_1} X_1, \nabla^M_{U_i} U_i \rangle^M &= \sum_{i=1}^{m} \langle \nabla^M_{X_1} X_1, X_2 \rangle^M \langle \nabla^M_{U_i} U_i, X_2 \rangle^M \\ &+ \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \langle \nabla^M_{X_1} X_1, U_j \rangle^M \langle \nabla^M_{U_i} U_i, U_j \rangle^M\end{aligned}$$

which completes the proof.

(4) Similar to Part (3).

(5) The proof follows from the following relation and the fact that $[U_i, U_j]$ is vertical.

$$\langle \nabla^M_{U_i} U_j, X_k \rangle^M = \langle \nabla^M_{U_j} U_i, X_k \rangle^M + \langle [U_i, U_j], X_k \rangle^M.$$

□

Lemma 4.3. *For each* $i = 1, \ldots, m$,

(1) $\langle [X_1, U_i], X_1 \rangle^M = -\frac{U_i(\lambda_1)}{2\lambda_1}$.

(2) $\langle [X_2, U_i], X_2 \rangle^M = -\frac{U_i(\lambda_2)}{2\lambda_2}$.

(3) $\langle [X_1, U_i], X_2 \rangle^M = -\frac{1}{\lambda_2} \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N$.

(4) $\langle [X_2, U_i], X_1 \rangle^M = \frac{1}{\lambda_1} \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N$.

Proof. For each $i = 1, \ldots, m$,

(1)

$$\begin{aligned}
\langle [X_1, U_i], X_1 \rangle^M &= \frac{1}{\lambda_1} \phi^* \langle [X_1, U_i], X_1 \rangle^N \\
&= -\frac{1}{\lambda_1} \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_1 \rangle^N \quad \text{By Lemma 4.2 (1)} \\
&= -\frac{U_i(\lambda_1)}{2\lambda_1} \quad \text{By Lemma 4.2 (2).}
\end{aligned}$$

Parts (2), (3), (4) are similar to Part (1). □

Lemma 4.4. *For each* $i = 1, \ldots, m$,

(1) $\langle \nabla^M_{X_1} X_1, U_i \rangle^M = \frac{U_i(\lambda_1)}{2\lambda_1}$.

(2) $\langle \nabla^M_{X_2} X_2, U_i \rangle^M = \frac{U_i(\lambda_2)}{2\lambda_2}$.

(3) $\langle \nabla^M_{X_2} X_1, U_i \rangle^M = -\frac{1}{\lambda_1} \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N + \langle \nabla^M_{U_i} X_1, X_2 \rangle^M$.

(4) $\langle \nabla^M_{X_1} X_2, U_i \rangle^M = \frac{1}{\lambda_2} \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N - \langle \nabla^M_{U_i} X_1, X_2 \rangle^M$.

Proof. For each $i = 1, \ldots, m$,

(1)

$$\begin{aligned}
\langle \nabla^M_{X_1} X_1, U_i \rangle^M &= -\langle \nabla^M_{X_1} U_i, X_1 \rangle^M \\
&= -\left\{ \langle \nabla^M_{U_i} X_1, X_1 \rangle^M + \langle [X_1, U_i], X_1 \rangle^M \right\} \\
&= \frac{U_i(\lambda_1)}{2\lambda_1} \quad \text{By Lemma 4.3 (1).}
\end{aligned}$$

(2) Similar to Part (1).

(3)

$$\begin{aligned}\langle \nabla^M_{X_2} X_1, U_i\rangle^M &= -\langle \nabla^M_{X_2} U_i, X_1\rangle^M \\ &= -\left\{\langle \nabla^M_{U_i} X_2, X_1\rangle^M + \langle [X_2, U_i], X_1\rangle^M\right\} \\ &= \langle \nabla^M_{U_i} X_1, X_2\rangle^M - \underbrace{\frac{1}{\lambda_1}\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N}_{\text{By Lemma 4.3 (4).}}\end{aligned}$$

(4) Similar to Part (3).

$\square$

Lemma 4.5. *For each* $i = 1, \ldots, m$,

(1)

$$\begin{aligned}\langle \nabla^M_{[X_1,U_i]} U_i, X_1\rangle^M &= \frac{(U_i(\lambda_1))^2}{4{\lambda_1}^2} + \frac{1}{\lambda_2}\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N \langle \nabla^M_{U_i} X_1, X_2\rangle^M \\ &- \frac{1}{\lambda_1\lambda_2}\left\{\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\right\}^2 + \sum_{j=1}^m (\langle \nabla^M_{U_i} U_j, X_1\rangle^M)^2 \\ &+ \sum_{j=1, j\neq i}^m A_{ij}B_{ij}\end{aligned}$$

where $A_{ij}B_{ij} = \langle \nabla^M_{X_1} U_i, U_j\rangle^M \langle \nabla^M_{U_i} U_j, X_1\rangle^M$.

(2)

$$\begin{aligned}\langle \nabla^M_{[X_2,U_i]} U_i, X_2\rangle^M &= \frac{(U_i(\lambda_2))^2}{4{\lambda_2}^2} + \frac{1}{\lambda_1}\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N \langle \nabla^M_{U_i} X_1, X_2\rangle^M \\ &- \frac{1}{\lambda_1\lambda_2}\left\{\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\right\}^2 + \sum_{j=1}^m (\langle \nabla^M_{U_i} U_j, X_2\rangle^M)^2 \\ &+ \sum_{j=1, j\neq i}^m A'_{ij}B'_{ij}\end{aligned}$$

where $A'_{ij}B'_{ij} = \langle \nabla^M_{X_2} U_i, U_j\rangle^M \langle \nabla^M_{U_i} U_j, X_2\rangle^M$.

Proof. Let $i = 1, \ldots, m$.

(1) Writing $[X_1, U_i]$ in terms of basis $(X_1, X_2, U_i)_{i=1}^m$, we have

$$\begin{aligned}
\langle\nabla^M_{[X_1,U_i]}U_i, X_1\rangle^M &= \langle[X_1, U_i], X_1\rangle^M \langle\nabla^M_{X_1}U_i, X_1\rangle^M + \langle[X_1, U_i], X_2\rangle^M \langle\nabla^M_{X_2}U_i, X_1\rangle^M \\
&+ \sum_{j=1}^m \langle[X_1, U_i], U_j\rangle^M \langle\nabla^M_{U_j}U_i, X_1\rangle^M \\
&= -\langle[X_1, U_i], X_1\rangle^M \langle\nabla^M_{X_1}X_1, U_i\rangle^M - \langle[X_1, U_i], X_2\rangle^M \langle\nabla^M_{X_2}X_1, U_i\rangle^M \\
&+ \sum_{j=1}^m \langle[X_1, U_i], U_j\rangle^M \langle\nabla^M_{U_j}U_i, X_1\rangle^M. \qquad (4.1)
\end{aligned}$$

Using $\langle[X_1, U_i], U_i\rangle^M = \langle\nabla^M_{X_1}U_i, U_i\rangle^M - \langle\nabla^M_{U_i}X_1, U_i\rangle^M = \langle\nabla^M_{U_i}U_i, X_1\rangle^M$, $\langle[X_1, U_i], U_j\rangle^M = \langle\nabla^M_{X_1}U_i, U_j\rangle^M + \langle\nabla^M_{U_i}U_j, X_1\rangle^M$ and Lemma 4.2 (5), the last term in Equation 4.1 can be simplified as follows.

$$\begin{aligned}
\sum_{j=1}^m \langle[X_1, U_i], U_j\rangle^M \langle\nabla^M_{U_j}U_i, X_1\rangle^M &= \sum_{i=1}^m \langle[X_1, U_i], U_i\rangle^M \langle\nabla^M_{U_i}U_i, X_1\rangle^M \\
&+ \sum_{j=1, j\neq i}^m \langle[X_1, U_i], U_j\rangle^M \langle\nabla^M_{U_j}U_i, X_1\rangle^M \\
&= \sum_{i=1}^m (\langle\nabla^M_{U_i}U_i, X_1\rangle^M)^2 + \sum_{j=1, j\neq i}^m (\langle\nabla^M_{U_i}U_j, X_1\rangle^M)^2 \\
&+ \sum_{j=1, j\neq i}^m \langle\nabla^M_{X_1}U_i, U_j\rangle^M \langle\nabla^M_{U_i}U_j, X_1\rangle^M. \qquad (4.2)
\end{aligned}$$

The proof is completed by substituting Equation 4.2 in Equation 4.1 and using Lemma 4.3 (1),(2) and Lemma 4.4 (1),(2) to determine the first two terms of Equation 4.1.

(2) Similar to Part (2).

□

Lemma 4.6. *For each $i = 1, \ldots, m$,*

(1)

$$\begin{aligned}
\langle\nabla^M_{U_i}\nabla^M_{X_1}U_i, X_1\rangle^M &= -U_i\left(\frac{U_i(\lambda_1)}{2\lambda_1}\right) + \frac{1}{\lambda_2}\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N \langle\nabla^M_{U_i}X_1, X_2\rangle^M \\
&- (\langle\nabla^M_{U_i}X_1, X_2\rangle^M)^2 + \sum_{j=1, j\neq i}^m A_{ij}B_{ij}
\end{aligned}$$

where $A_{ij}B_{ij} = \langle \nabla^M_{X_1} U_i, U_j\rangle^M \langle \nabla^M_{U_i} U_j, X_1\rangle^M$.

(2)

$$\begin{aligned}\langle \nabla^M_{U_i}\nabla^M_{X_2} U_i, X_2\rangle^M &= -U_i\left(\frac{U_i(\lambda_2)}{2\lambda_2}\right) + \frac{1}{\lambda_1}\langle \nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N \langle \nabla^M_{U_i} X_1, X_2\rangle^M \\ &- (\langle \nabla^M_{U_i} X_1, X_2\rangle^M)^2 + \sum_{j=1, j\neq i}^m A'_{ij}B'_{ij}\end{aligned}$$

where $A'_{ij}B'_{ij} = \langle \nabla^M_{X_2} U_i, U_j\rangle^M \langle \nabla^M_{U_i} U_j, X_2\rangle^M$.

Proof. (1)

$$\begin{aligned}\langle \nabla^M_{U_i}\nabla^M_{X_1} U_i, X_1\rangle^M &= U_i(\langle \nabla^M_{X_1} U_i, X_1\rangle^M) - \langle \nabla^M_{X_1} U_i, \nabla^M_{U_i} X_1\rangle^M \\ &= -U_i(\langle \nabla^M_{X_1} X_1, U_i\rangle^M) - \langle \nabla^M_{X_1} U_i, \nabla^M_{U_i} X_1\rangle^M.\end{aligned}$$

Now $\nabla^M_{X_1} U_i = \langle \nabla^M_{X_1} U_i, X_1\rangle^M X_1 + \langle \nabla^M_{X_1} U_i, X_2\rangle^M X_2 + \sum_{j=1, j\neq i}^m \langle \nabla^M_{X_1} U_i, U_j\rangle^M U_j$. Therefore,

$$\begin{aligned}\langle \nabla^M_{U_i}\nabla^M_{X_1} U_i, X_1\rangle^M &= -U_i(\langle \nabla^M_{X_1} X_1, U_i\rangle^M) - \langle \nabla^M_{X_1} U_i, X_2\rangle^M \langle \nabla^M_{U_i} X_1, X_2\rangle^M \\ &- \sum_{j=1, j\neq i}^m \langle \nabla^M_{X_1} U_i, U_j\rangle^M \langle \nabla^M_{U_i} X_1, U_j\rangle^M \\ &= -U_i(\langle \nabla^M_{X_1} X_1, U_i\rangle^M) + \langle \nabla^M_{X_1} X_2, U_i\rangle^M \langle \nabla^M_{U_i} X_1, X_2\rangle^M \\ &+ \sum_{j=1, j\neq i}^m \langle \nabla^M_{X_1} U_i, U_j\rangle^M \langle \nabla^M_{U_i} U_j, X_1\rangle^M.\end{aligned}$$

Using Lemma 4.4 (1),(4) for the first two terms in the above expression gives the required result.

(2) Similar to Part (1).

□

Let K_l denote the Ricci curvature given by $K_l = \sum_{i=1}^m \langle R^M(X_l, U_i)X_l, U_i\rangle^M$.

Lemma 4.7. *The Ricci curvatures* K_l, $l = 1, 2$ *are given by*

(1)

$$\begin{aligned}
\lambda_1 K_1 &= -\sum_{i=1}^{m}\left[-\frac{1}{2}U_i\left(U_i(\lambda_1)\right)+\frac{3}{4}\frac{(U_i(\lambda_1))^2}{\lambda_1}-\frac{1}{\lambda_2}\left\{\langle\nabla_{U_i}^{\phi^{-1}TN}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\right\}^2\right]\\
&- 2\sum_{i=1}^{m}\frac{\lambda_1}{\lambda_2}\langle\nabla_{U_i}^{\phi^{-1}TN}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M\\
&- \sum_{i=1}^{m}\left[-\lambda_1(\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M)^2+\lambda_1\sum_{j=1}^{m}(\langle\nabla_{U_i}^{M}U_j, X_1\rangle^M)^2\right]+\lambda_1\sum_{i=1}^{m}\langle\nabla_{X_1}^{M}\nabla_{U_i}^{M}U_i, X_1\rangle^M.
\end{aligned}$$

(2)

$$\begin{aligned}
\lambda_2 K_2 &= -\sum_{i=1}^{m}\left[-\frac{1}{2}U_i\left(U_i(\lambda_2)\right)+\frac{3}{4}\frac{(U_i(\lambda_2))^2}{\lambda_2}-\frac{1}{\lambda_1}\left\{\langle\nabla_{U_i}^{\phi^{-1}TN}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\right\}^2\right]\\
&- 2\sum_{i=1}^{m}\frac{\lambda_2}{\lambda_1}\langle\nabla_{U_i}^{\phi^{-1}TN}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M\\
&- \sum_{i=1}^{m}\left[-\lambda_2(\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M)^2+\lambda_2\sum_{j=1}^{m}(\langle\nabla_{U_i}^{M}U_j, X_1\rangle^M)^2\right]+\lambda_2\sum_{i=1}^{m}\langle\nabla_{X_2}^{M}\nabla_{U_i}^{M}U_i, X_2\rangle^M.
\end{aligned}$$

Proof. (1) By definition

$$\begin{aligned}
K_1 &= \sum_{i=1}^{m}\langle R^M(X_1,U_i)X_1, U_i\rangle^M = -\sum_{i=1}^{m}\langle R^M(X_1,U_i)U_i, X_1\rangle^M\\
&= -\sum_{i=1}^{m}\left\{\underline{\langle\nabla_{U_i}^{M}\nabla_{X_1}^{M}U_i, X_1\rangle^M}+\langle\nabla_{[X_1,U_i]}^{M}U_i, X_1\rangle^M\right\}+\sum_{i=1}^{m}\langle\nabla_{X_1}^{M}\nabla_{U_i}^{M}U_i, X_1\rangle^M. \qquad (4.3)
\end{aligned}$$

We calculate the underlined term in the above expression using Lemma 4.6 (1).

$$\begin{aligned}
\sum_{i=1}^{m}\langle\nabla_{U_i}^{M}\nabla_{X_1}^{M}U_i, X_1\rangle^M &= \sum_{i=1}^{m}\left\{-U_i\left(\frac{U_i(\lambda_1)}{2\lambda_1}\right)-(\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M)^2\right\}\\
&+ \sum_{i=1}^{m}\frac{1}{\lambda_2}\langle\nabla_{U_i}^{\phi^{-1}TN}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\langle\nabla_{U_i}^{M}X_1, X_2\rangle^M+\sum_{i,j=1}^{m}A_{ij}B_{ij}
\end{aligned}$$

where $A_{ij}=\langle\nabla_{X_1}^{M}U_i, U_j\rangle^M$ and $B_{ij}=\langle\nabla_{U_i}^{M}U_j, X_1\rangle^M$.

Since A_{ij} is antisymmetric in i,j and B_{ij} is symmetric in i,j, we note that

$$\sum_{i,j=1}^{m}A_{ij}B_{ij}=0. \qquad (4.4)$$

Moreover, $-U_i\left(\frac{U_i(\lambda_1)}{2\lambda_1}\right) = -\frac{1}{2}\left\{\frac{1}{\lambda_1}U_i\left(U_i(\lambda_1)\right) - \frac{(U_i(\lambda_1))^2}{{\lambda_1}^2}\right\}$. Using this and Equation 4.4 we obtain

$$\begin{aligned}\sum_{i=1}^{m}\langle\nabla^M_{U_i}\nabla^M_{X_1}U_i, X_1\rangle^M &= \sum_{i=1}^{m}\left\{-\frac{1}{2\lambda_1}U_i\left(U_i(\lambda_1)\right) + \frac{(U_i(\lambda_1))^2}{2{\lambda_1}^2} - (\langle\nabla^M_{U_i}X_1, X_2\rangle^M)^2\right\}\\ &+ \sum_{i=1}^{m}\frac{1}{\lambda_2}\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\langle\nabla^M_{U_i}X_1, X_2\rangle^M. \qquad (4.5)\end{aligned}$$

Substituting the Equation 4.5 and expression for $\langle\nabla^M_{[X_1,U_i]}U_i, X_1\rangle^M$ from Lemma 4.5 (1) into Equation 4.3 and using $\sum_{i,j=1}^{m}A_{ij}B_{ij} = 0$ gives the required expression for K_1.

(2) Similar to Part (1).

□

The expression for the vertical Laplacian is given by the following.

Proposition 4.8 (Vertical Laplacian)**.**

$$\begin{aligned}-\frac{1}{2}\Delta^v\mu &= \sum_{i=1}^{m}\left[\lambda_1K_1 - \lambda_2K_2 + \frac{3}{4}\left\{\frac{(U_i(\lambda_1))^2}{\lambda_1} - \frac{(U_i(\lambda_2))^2}{\lambda_2}\right\}\right]\\ &+ 2\sum_{i=1}^{m}\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\langle\nabla^M_{U_i}X_1, X_2\rangle^M\\ &- \sum_{i=1}^{m}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\left\{\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\right\}^2\\ &+ \sum_{i=1}^{m}\left\{-\mu(\langle\nabla^M_{U_i}X_1, X_2\rangle^M)^2 + \sum_{j=1}^{m}\left(\lambda_1(\langle\nabla^M_{U_i}U_j, X_1\rangle^M)^2 - \lambda_2(\langle\nabla^M_{U_i}U_j, X_2\rangle^M)^2\right)\right\}\\ &+ \sum_{i=1}^{m}\left\{-\lambda_1\langle\nabla^M_{X_1}\nabla^M_{U_i}U_i, X_1\rangle^M + \lambda_2\langle\nabla^M_{X_2}\nabla^M_{U_i}U_i, X_2\rangle^M\right\} - \sum_{i=1}^{m}\frac{1}{2}d\mu\left(\nabla^M_{U_i}U_i\right). \qquad (4.6)\end{aligned}$$

Proof.

$$\begin{aligned}-\frac{1}{2}\Delta^v\mu &= \frac{1}{2}\sum_{i=1}^{m}\nabla d\mu(U_i, U_i)\\ &= \frac{1}{2}\sum_{i=1}^{m}U_i\left(U_i(\mu)\right) - \sum_{i=1}^{m}\frac{1}{2}d\mu\left(\nabla^M_{U_i}U_i\right)\end{aligned}$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left\{U_i\left(U_i(\lambda_1)\right) - U_i\left(U_i(\lambda_2)\right)\right\} - \sum_{i=1}^{m}\frac{1}{2}d\mu\left(\nabla^M_{U_i}U_i\right). \tag{4.7}$$

The proof is completed by substituting the expressions from Lemma 4.7 (1), (2) for $\frac{1}{2}\sum_{i=1}^{m}U_i\left(U_i(\lambda_1)\right)$ and $\frac{1}{2}\sum_{i=1}^{m}U_i\left(U_i(\lambda_2)\right)$ into Equation 4.7.

□

4.2. **The horizontal Laplacian.** The horizontal Laplacian is given by

$$-\Delta^h\mu = \sum_{k=1}^{2}\left\{\nabla_{X_k}d\mu(X_k) - d\mu\left(\nabla^M_{X_k}X_k\right)\right\}.$$

So, firstly, we calculate $\nabla_{X_k}d\mu(X_k)$ and $d\mu\left(\nabla^M_{X_k}X_k\right)$ in Lemma 4.11 and Lemma 4.10 respectively, with the help of following Lemma.

Lemma 4.9.

(1) $d\mu(X_1) = 2\mu\langle\nabla^M_{X_2}X_2, X_1\rangle^M + 2\lambda_1\sum_{i=1}^{m}\langle\nabla^M_{U_i}U_i, X_1\rangle^M.$

(2) $d\mu(X_2) = 2\mu\langle\nabla^M_{X_1}X_1, X_2\rangle^M - 2\lambda_2\sum_{i=1}^{m}\langle\nabla^M_{U_i}U_i, X_2\rangle^M.$

Proof. (1) Let S be the stress energy tensor of the map ϕ. Recall from [4] that the stress energy tensor of a smooth map $\phi : (M, \langle\cdot,\cdot\rangle^M) \to (N, \langle\cdot,\cdot\rangle^N)$ is given by $S = \frac{1}{2}\|d\phi\|^2 \cdot \langle\cdot,\cdot\rangle^M - \phi^*\langle\cdot,\cdot\rangle^N$. Moreover, if ϕ is harmonic , then $\mathbf{div}S = 0$. Therefore,

$$(\mathbf{div}S)(X_1) = 0 = \sum_{k=1}^{2}\left(\nabla_{X_k}S\right)(X_k, X_1) + \sum_{i=1}^{m}\left(\nabla_{U_i}S\right)(U_i, X_1). \tag{4.8}$$

Now

$$\begin{aligned}
\left(\nabla_{X_1}S\right)(X_1, X_1) &= \frac{1}{2}X_1\left(\|d\phi\|^2\right)\cdot\langle X_1, X_1\rangle^M \\
&- \left\{X_1\left(\phi^*\langle X_1, X_1\rangle^N\right) - 2\phi^*\langle\nabla^M_{X_1}X_1, X_1\rangle^N\right\} \\
&= \frac{1}{2}X_1\left(\lambda_1+\lambda_2\right) - \left\{X_1\left(\lambda_1\langle X_1, X_1\rangle^M\right) - 2\lambda_1\langle\nabla^M_{X_1}X_1, X_1\rangle^M\right\} \\
&= -\frac{1}{2}X_1(\mu) = -\frac{1}{2}d\mu(X_1), \qquad (4.9)
\end{aligned}$$

$$
\begin{aligned}
(\nabla_{X_2} S)(X_2, X_1) &= \frac{1}{2} X_2 \left(\|d\phi\|^2\right) \cdot \langle X_2, X_1 \rangle^M \\
&- \left\{ X_2 \left(\phi^* \langle X_2, X_1 \rangle^N \right) - \phi^* \langle \nabla^M_{X_2} X_2, X_1 \rangle^N - \phi^* \langle \nabla^M_{X_2} X_1, X_2 \rangle^N \right\} \\
&= -\left\{ X_2 \left(\lambda_1 \langle X_2, X_1 \rangle^M \right) - \lambda_1 \langle \nabla^M_{X_2} X_2, X_1 \rangle^M - \lambda_2 \langle \nabla^M_{X_2} X_1, X_2 \rangle^M \right\} \\
&= \mu \langle \nabla^M_{X_2} X_2, X_1 \rangle^M. \qquad (4.10)
\end{aligned}
$$

Similarly

$$
(\nabla_{U_i} S)(U_i, X_1) = \lambda_1 \langle \nabla^M_{U_i} U_i, X_1 \rangle^M. \qquad (4.11)
$$

Equation 4.8 combined with Equations 4.9, 4.10 and 4.11 completes the proof.

(2) Considering $(\mathbf{div} S)(X_2) = 0$ and proceeding as in Part (1) gives the proof.

□

Lemma 4.10.

(1)

$$
\begin{aligned}
d\mu\left(\nabla^M_{X_1} X_1\right) &= 2\mu(\langle \nabla^M_{X_1} X_1, X_2 \rangle^M)^2 - 2\lambda_2 \sum_{i=1}^m \langle \nabla^M_{X_1} X_1, \nabla^M_{U_i} U_i \rangle^M \\
&+ 2\lambda_2 \sum_{i=1}^m \sum_{j=1, j\neq i}^m \langle \nabla^M_{X_1} X_1, U_j \rangle^M \langle \nabla^M_{U_i} U_i, U_j \rangle^M \\
&+ \sum_{i=1}^m \frac{U_i(\lambda_1)}{2\lambda_1} U_i(\mu).
\end{aligned}
$$

(2)

$$
\begin{aligned}
d\mu\left(\nabla^M_{X_2} X_2\right) &= 2\mu(\langle \nabla^M_{X_2} X_2, X_1 \rangle^M)^2 + 2\lambda_1 \sum_{i=1}^m \langle \nabla^M_{X_2} X_2, \nabla^M_{U_i} U_i \rangle^M \\
&- 2\lambda_1 \sum_{i=1}^m \sum_{j=1, j\neq i}^m \langle \nabla^M_{X_2} X_2, U_j \rangle^M \langle \nabla^M_{U_i} U_i, U_j \rangle^M \\
&+ \sum_{i=1}^m \frac{U_i(\lambda_2)}{2\lambda_2} U_i(\mu).
\end{aligned}
$$

Proof. (1) Writing $\nabla^M_{X_1} X_1 = \langle \nabla^M_{X_1} X_1, X_2 \rangle^M X_2 + \sum_{i=1}^m \langle \nabla^M_{X_1} X_1, U_i \rangle^M U_i$ we have

$$
d\mu\left(\nabla^M_{X_1} X_1\right) = \langle \nabla^M_{X_1} X_1, X_2 \rangle^M d\mu(X_2) + \sum_{i=1}^m \langle \nabla^M_{X_1} X_1, U_i \rangle^M d\mu(U_i).
$$

Using Lemma 4.9 (2) and Lemma 4.4 (1) we obtain

$$\begin{aligned} d\mu\left(\nabla^M_{X_1}X_1\right) &= 2\mu(\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2 - 2\lambda_2\sum_{i=1}^{m}\langle\nabla^M_{X_1}X_1, X_2\rangle^M\langle\nabla^M_{U_i}U_i, X_2\rangle^M \\ &+ \sum_{i=1}^{m}\frac{U_i(\lambda_1)}{2\lambda_1}U_i(\mu). \end{aligned}$$

The proof finishes after simplifying further, using Lemma 4.2 (3).

(2) Similar to Part (1).

□

Lemma 4.11.

(1)

$$\begin{aligned} \nabla_{X_1}d\mu(X_1) &= 4\mu(\langle\nabla^M_{X_2}X_2, X_1\rangle^M)^2 + 4\lambda_1\sum_{i=1}^{m}\langle\nabla^M_{X_2}X_2, \nabla^M_{U_i}U_i\rangle^M \\ &- 4\lambda_1\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\langle\nabla^M_{X_2}X_2, U_j\rangle^M\langle\nabla^M_{U_i}U_i, U_j\rangle^M \\ &+ 2\mu\left\{\langle\nabla^M_{X_1}\nabla^M_{X_2}X_2, X_1\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M\right\} \\ &+ 2\sum_{i=1}^{m}X_1\left(\lambda_1\langle\nabla^M_{U_i}U_i, X_1\rangle^M\right). \end{aligned}$$

(2)

$$\begin{aligned} \nabla_{X_2}d\mu(X_2) &= 4\mu(\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2 - 4\lambda_2\sum_{i=1}^{m}\langle\nabla^M_{X_1}X_1, \nabla^M_{U_i}U_i\rangle^M \\ &+ 4\lambda_2\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\langle\nabla^M_{X_1}X_1, U_j\rangle^M\langle\nabla^M_{U_i}U_i, U_j\rangle^M \\ &+ 2\mu\left\{\langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M\right\} \\ &- 2\sum_{i=1}^{m}X_2\left(\lambda_2\langle\nabla^M_{U_i}U_i, X_2\rangle^M\right). \end{aligned}$$

Proof. (1) Using Lemma 4.9 (1) we can write

$$
\begin{aligned}
\nabla_{X_1} d\mu(X_1) &= 2X_1(\mu)\cdot\langle\nabla^M_{X_2}X_2, X_1\rangle^M + 2\mu\cdot X_1\left(\langle\nabla^M_{X_2}X_2, X_1\rangle^M\right)\\
&+ 2\sum_{i=1}^m X_1\left(\lambda_1\langle\nabla^M_{U_i}U_i, X_1\rangle^M\right)\\
&= 2d\mu(X_1)\cdot\langle\nabla^M_{X_2}X_2, X_1\rangle^M\\
&+ 2\mu\left\{\langle\nabla^M_{X_1}\nabla^M_{X_2}X_2, X_1\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M\right\}\\
&+ 2\sum_{i=1}^m X_1\left(\lambda_1\langle\nabla^M_{U_i}U_i, X_1\rangle^M\right).
\end{aligned}
$$

Substituting the value of $d\mu(X_1)$ from Lemma 4.9 (1) and using Lemma 4.2 (4) gives the required relation.

(2) Similar to Part (1).

□

The first version of the formula for the horizontal Laplacian is

Proposition 4.12.

$$
\begin{aligned}
-\Delta^h\mu &= -\frac{1}{2}\sum_{i=1}^m U_i(\mu)\left(\frac{U_i(\lambda_1)}{\lambda_1}+\frac{U_i(\lambda_2)}{\lambda_2}\right) + 2\mu\left\{(\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2 + (\langle\nabla^M_{X_2}X_2, X_1\rangle^M)^2\right\}\\
&+ 2\mu\left\{\langle\nabla^M_{X_1}\nabla^M_{X_2}X_2, X_1\rangle^M + 2\langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M + \langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M\right\}\\
&+ \sum_{i=1}^m\left\{2\lambda_1\langle\nabla^M_{X_2}X_2, \nabla^M_{U_i}U_i\rangle^M - 2\lambda_2\langle\nabla^M_{X_1}X_1, \nabla^M_{U_i}U_i\rangle^M\right\}\\
&+ \sum_{i=1}^m\sum_{j=1, j\neq i}^m\langle\nabla^M_{U_i}U_i, U_j\rangle^M\left\{-2\lambda_1\langle\nabla^M_{X_2}X_2, U_j\rangle^M + 2\lambda_2\langle\nabla^M_{X_1}X_1, U_j\rangle^M\right\}\\
&+ 2\sum_{i=1}^m\left\{X_1\left(\lambda_1\langle\nabla^M_{U_i}U_i, X_1\rangle^M\right) - 2X_2\left(\lambda_2\langle\nabla^M_{U_i}U_i, X_2\rangle^M\right)\right\}.
\end{aligned}
$$

Proof. According to definition the horizontal Laplacian $\Delta^h\mu$ is

$$
-\Delta^h\mu = \sum_{k=1}^2\nabla d\mu(X_k, X_k) = \sum_{k=1}^2\left\{\nabla_{X_k}d\mu(X_k) - d\mu\left(\nabla^M_{X_k}X_k\right)\right\}.
$$

Calculating $\sum_{k=1}^2 d\mu\left(\nabla^M_{X_k}X_k\right)$ from Lemma 4.10 and $\sum_{k=1}^2\nabla_{X_k}d\mu(X_k)$ from Lemma 4.11 gives the expression for the horizontal Laplacian. □

Before giving the final expression for the horizontal Laplacian we simplify, in Lemma 4.13 and Lemma 4.14, some of the terms present in Proposition 4.12.

Lemma 4.13.

(1)

$$\begin{aligned}&\langle\nabla^M_{X_1}\nabla^M_{X_2}X_2, X_1\rangle^M + 2\langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M + \langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M \\ =\;&\langle R^M(X_1,X_2)X_1, X_2\rangle^M + \langle\nabla^M_{[X_1,X_2]}X_2, X_1\rangle^M + \sum_{i=1}^m \frac{U_i(\lambda_1)\cdot U_i(\lambda_2)}{4\lambda_1\lambda_2} \\ &+\sum_{i=1}^m[\{\langle\nabla^M_{U_i}X_1, X_2\rangle^M - \frac{1}{\lambda_1}\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\} \\ &\times\{\langle\nabla^M_{U_i}X_1, X_2\rangle^M - \frac{1}{\lambda_2}\langle\nabla^{\phi^{-1}TN}_{U_i}d\phi\cdot X_1, d\phi\cdot X_2\rangle^N\}].\end{aligned}$$

(2)

$$\begin{aligned}&\sum_{i=1}^m\sum_{j=1,j\neq i}^m \langle\nabla^M_{U_i}U_i, U_j\rangle^M\left\{-2\lambda_1\langle\nabla^M_{X_2}X_2, U_j\rangle^M + 2\lambda_2\langle\nabla^M_{X_1}X_1, U_j\rangle^M\right\} \\ =\;&\sum_{i=1}^m\sum_{j=1,j\neq i}^m \langle\nabla^M_{U_i}U_i, U_j\rangle^M\left\{-\frac{\lambda_1}{\lambda_2}U_j(\lambda_2) + \frac{\lambda_2}{\lambda_1}U_j(\lambda_1)\right\}.\end{aligned}$$

Proof. (1) By definition $\nabla^M_{X_1}\nabla^M_{X_2}X_2 = -R^M(X_1,X_2)X_2 + \nabla^M_{X_2}\nabla^M_{X_1}X_2 + \nabla^M_{[X_1,X_2]}X_2$. Therefore, we can write as

$$\begin{aligned}&\langle\nabla^M_{X_1}\nabla^M_{X_2}X_2, X_1\rangle^M + 2\langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M + \langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M \\ =\;&\langle R^M(X_1,X_2)X_1, X_2\rangle^M + \langle\nabla^M_{[X_1,X_2]}X_2, X_1\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M \\ &+\left\{\langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M + \langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M\right\}\end{aligned}$$

(4.12)

where the terms in the braces can be simplified as follows.

$$\begin{aligned} & \langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M + \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M + \langle\nabla^M_{X_2}\nabla^M_{X_1}X_1, X_2\rangle^M \\ = & \\ & \langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M + X_2\left(\langle\nabla^M_{X_1}X_1, X_2\rangle^M\right) \\ & \langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M - X_2\left(\langle\nabla^M_{X_1}X_2, X_1\rangle^M\right) \\ & \langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M - \langle\nabla^M_{X_2}\nabla^M_{X_1}X_2, X_1\rangle^M - \langle\nabla^M_{X_1}X_2, \nabla^M_{X_2}X_1\rangle^M \\ = & -\sum_{i=1}^m \langle\nabla^M_{X_2}X_1, U_i\rangle^M \langle\nabla^M_{X_1}X_2, U_i\rangle^M \end{aligned} \tag{4.13}$$

where we have used $\nabla^M_{X_2}X_1 = \langle\nabla^M_{X_2}X_1, X_2\rangle^M X_2 + \sum_{i=1}^m \langle\nabla^M_{X_2}X_1, U_i\rangle^M U_i$.

Moreover,

$$\begin{aligned} \langle\nabla^M_{X_1}X_1, \nabla^M_{X_2}X_2\rangle^M &= \sum_{i=1}^m \langle\nabla^M_{X_1}X_1, U_i\rangle^M \langle\nabla^M_{X_2}X_2, U_i\rangle^M \\ &= \sum_{i=1}^m \frac{U_i(\lambda_1)\cdot U_i(\lambda_2)}{4\lambda_1\lambda_2}. \end{aligned} \tag{4.14}$$

The proof is completed by substituting Equation 4.13, Equation 4.14 in Equation 4.12 and using Lemma 4.4 (3),(4).

(2) Follows directly from Lemma 4.4 (1),(2).

□

Lemma 4.14.

$$\begin{aligned} \langle\nabla^M_{[X_1,X_2]}X_2, X_1\rangle^M &= (\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2 + (\langle\nabla^M_{X_2}X_2, X_1\rangle^M)^2 \\ &- \sum_{i=1}^m \left(\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\right) \langle\nabla^{\phi^{-1}TN}_{U_i} d\phi\cdot X_1, d\phi\cdot X_2\rangle^N \langle\nabla^M_{U_i}X_1, X_2\rangle^M \\ &+ 2\sum_{i=1}^m \langle\nabla^M_{U_i}X_1, X_2\rangle^M \langle\nabla^M_{U_i}X_1, X_2\rangle^M. \end{aligned}$$

Proof. Writing $[X_1, X_2] = \sum_{k=1}^{2} \langle [X_1, X_2], X_k \rangle^M X_k + \sum_{i=1}^{m} \langle [X_1, X_2], U_i \rangle^M U_i$ we obtain

$$\begin{aligned}
\langle \nabla^M_{[X_1,X_2]} X_2, X_1 \rangle^M &= \langle [X_1, X_2], X_1 \rangle^M \langle \nabla^M_{X_1} X_2, X_1 \rangle^M + \langle [X_1, X_2], X_2 \rangle^M \langle \nabla^M_{X_2} X_2, X_1 \rangle^M \\
&- \sum_{i=1}^{m} \langle [X_1, X_2], U_i \rangle^M \langle \nabla^M_{U_i} X_1, X_2 \rangle^M \\
&= \left\{ \langle \nabla^M_{X_1} X_2, X_1 \rangle^M - 0 \right\} \langle \nabla^M_{X_1} X_2, X_1 \rangle^M \\
&+ \left\{ 0 - \langle \nabla^M_{X_2} X_1, X_2 \rangle^M \right\} \langle \nabla^M_{X_2} X_2, X_1 \rangle^M \\
&- \sum_{i=1}^{m} \left\{ \langle \nabla^M_{X_1} X_2, U_i \rangle^M - \langle \nabla^M_{X_2} X_1, U_i \rangle^M \right\} \langle \nabla^M_{U_i} X_1, X_2 \rangle^M .
\end{aligned}$$

The desired relation is obtained by using Lemma 4.4 (3),(4). □

Collecting the above results we have:

Proposition 4.15 (Horizontal Laplacian)**.** *Writing* K_H *for* $\langle R^M(X_1, X_2)X_1, X_2 \rangle^M$ *we have*

$$\begin{aligned}
-\Delta^h \mu &= 2\mu K_H - \frac{1}{2} \sum_{i=1}^{m} \left\{ U_i(\mu) \left(\frac{U_i(\lambda_1)}{\lambda_1} + \frac{U_i(\lambda_2)}{\lambda_2} \right) + \mu \frac{U_i(\lambda_1) \cdot U_i(\lambda_2)}{\lambda_1 \lambda_2} \right\} \\
&+ 4\mu \left\{ (\langle \nabla^M_{X_1} X_1, X_2 \rangle^M)^2 + (\langle \nabla^M_{X_2} X_2, X_1 \rangle^M)^2 \right\} \\
&+ \sum_{i=1}^{m} \left\{ 6\mu (\langle \nabla^M_{U_i} X_1, X_2 \rangle^M)^2 + \frac{2\mu}{\lambda_1 \lambda_2} \left\{ \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N \right\}^2 \right\} \\
&- 4\mu \sum_{i=1}^{m} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \langle \nabla^{\phi^{-1}TN}_{U_i} d\phi \cdot X_1, d\phi \cdot X_2 \rangle^N \langle \nabla^M_{U_i} X_1, X_2 \rangle^M \\
&+ \sum_{i=1}^{m} \left\{ 2\lambda_1 \langle \nabla^M_{X_2} X_2, \nabla^M_{U_i} U_i \rangle^M - 2\lambda_2 \langle \nabla^M_{X_1} X_1, \nabla^M_{U_i} U_i \rangle^M \right\} \\
&+ \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} \langle \nabla^M_{U_i} U_i, U_j \rangle^M \left(-\frac{\lambda_1}{\lambda_2} U_j(\lambda_2) + \frac{\lambda_2}{\lambda_1} U_j(\lambda_1) \right) \\
&+ \sum_{i=1}^{m} \left\{ 2X_1 \left(\lambda_1 \langle \nabla^M_{U_i} U_i, X_1 \rangle^M \right) - 2X_2 \left(\lambda_2 \langle \nabla^M_{U_i} U_i, X_2 \rangle^M \right) \right\} .
\end{aligned} \tag{4.15}$$

Proof. The proof follows from Proposition 4.12, Lemma 4.13 and Lemma 4.14. □

5. Necessary conditions; Sufficiency and restrictions

In this section we present the required integral formula for $\Delta\mu^2$ for a harmonic map $\phi\colon (M^{m+2}, \langle\cdot,\cdot\rangle^M) \to (N^2, h)$ $(m \geq 1)$, which leads to the conditions making ϕ horizontally conformal. (See corollary 5.2). In Corollary 5.3, Corollary 5.4 we give restrictions on these conditions and finally Proposition 5.5 discusses the case when these necessary conditions are sufficient.

Theorem 5.1. *Let $m \geq 1$. Let $\phi\colon (M^{m+2}, \langle\cdot,\cdot\rangle^M) \to (N^2, \langle\cdot,\cdot\rangle^N)$ be a harmonic map which has rank 2 almost everywhere and μ be the function given by the difference of the non-trivial eigenvalues of the first fundamental form of ϕ, (as explained in Section 3). Then at each point $x \in M$,*

$$\begin{aligned}
-\Delta\mu^2 &= 4\mu\left\{\lambda_1 \mathbf{Ricci^M}(X_1,X_1) - \lambda_2 \mathbf{Ricci^M}(X_2,X_2)\right\} + 2\|d\mu\|^2 \\
&+ 8\mu^2 \|\mathcal{H}(\nabla^M_{X_1}X_1 + \nabla^M_{X_2}X_2)\|^2 + 4\mu\left\{a_1 X_1(\lambda_1) - a_2 X_2(\lambda_2)\right\} \\
&+ \sum_{i=1}^{m}\left[8\mu^2(\langle\nabla^M_{U_i}X_1, X_2\rangle^M)^2 + 2\mu\left\{\frac{(U_i(\lambda_1))^2}{\lambda_1} - \frac{(U_i(\lambda_2))^2}{\lambda_2}\right\}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1, j\neq i}^{m}\left\{\lambda_1(\langle\nabla^M_{U_i}U_j, X_1\rangle^M)^2 - \lambda_2(\langle\nabla^M_{U_i}U_j, X_2\rangle^M)^2\right\}
\end{aligned}$$

where $a_k = \sum_{i=1}^{m}\langle\nabla^M_{U_i}U_i, X_k\rangle^M$, for $k = 1,2$ *and $\mathcal{H}$ denotes the orthogonal projection on the horizontal space defined in Section 2.*

Proof. Follows from Proposition 4.1 and the fact that the following computational relations hold.

(1)

$$\begin{aligned}
\sum_{i=1}^{m} d\mu\left(\nabla^M_{U_i}U_i\right) &= 2\left\{\lambda_1 a_1^2 + \mu a_1 \cdot \langle\nabla^M_{X_2}X_2, X_1\rangle^M - \lambda_2 a_2^2 + \mu a_2 \cdot \langle\nabla^M_{X_1}X_1, X_2\rangle^M\right\} \\
&+ \sum_{i=1}^{m}\sum_{j=1, j\neq i}^{m}\langle\nabla^M_{U_i}U_i, U_j\rangle^M \cdot U_j(\mu).
\end{aligned}$$

(2) $\|\mathcal{H}(\nabla^M_{X_1}X_1 + \nabla^M_{X_2}X_2)\|^2 = (\langle\nabla^M_{X_2}X_2, X_1\rangle^M)^2 + (\langle\nabla^M_{X_1}X_1, X_2\rangle^M)^2.$

(3)

$$\begin{aligned}\sum_{i=1}^{m}\langle\nabla^M_{X_1}X_1+\nabla^M_{X_2}X_2,\nabla^M_{U_i}U_i\rangle^M &= a_1\langle\nabla^M_{X_2}X_2,X_1\rangle^M+a_2\langle\nabla^M_{X_1}X_1,X_2\rangle^M\\ &+\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\langle\nabla^M_{U_i}U_i,U_j\rangle^M\left\{\frac{U_j(\lambda_1)}{2\lambda_1}+\frac{U_j(\lambda_2)}{2\lambda_2}\right\}\end{aligned}$$

where $a_k=\sum_{i=1}^{m}\langle\nabla^M_{U_i}U_i,X_k\rangle^M$ for $k=1,2$. □

The following application of Theorem 5.1 provides the desired conditions.

Corollary 5.2. *Let $(M^{m+2},\langle\cdot,\cdot\rangle^M)$ be a closed Riemannian manifold. Let $\phi\colon M^{m+2}\to N^2$ be a harmonic map of rank 2 almost everywhere. Then ϕ is a harmonic morphism if the following conditions are satisfied:*

(1) $\mathbf{Ricci^M}(X_1,X_1)=\mathbf{Ricci^M}(X_2,X_2)>0$,
(2) *ϕ has totally geodesic fibres,*
(3) $\mathbf{grad}e(\phi)$ *is horizontal.*

Proof. Let x be a maximum point of μ^2 in the interior of M. Let $(U_i)_{i=1}^m$ be a basis of T_x^VM the vertical space at x. Then by Condition (3)

$$U_i(\lambda_1)(x)=-U_i(\lambda_2)(x). \tag{5.1}$$

On the other hand the maximality of μ^2 at x implies that $U_i(\mu^2)(x)=0$, i.e. either $\mu(x)=0$ or $U_i(\mu)(x)=0$. If $U_i(\mu)(x)=0$ then from Equation 5.1 we have

$$U_i(\lambda_1)(x)=0=U_i(\lambda_2)(x).$$

Moreover, the condition that ϕ has totally geodesic fibres implies that $\nabla^M_{U_i}U_j$ is vertical for all $i,j\in\{1,\dots,m\}$ hence $a_k=0$ for $k=1,2$. Now combining the hypothesis with Theorem 5.1 we have $-\Delta\mu^2>0$ at x, which is a contradiction to the maximality of μ^2 at x. Therefore $\mu(x)=0$ and hence we must have $\mu^2\equiv 0$. □

The restrictions on the conditions, determined above, are obtained by the application of the Weitzenböck formula for harmonic morphisms proved by the author in [9, Proposition 2.1].

Corollary 5.3. *There exists no non-constant harmonic map* $\phi\colon M^{m+2}\to N^2$*, of rank 2 almost everywhere, from a closed Riemannian manifold such that* N^2 *is a compact Riemann surface of genus* $g \geq 1$ *and the following conditions are satisfied:*

(1) $\mathbf{Ricci}(X_1, X_1) = \mathbf{Ricci}(X_2, X_2) > 0$
(2) ϕ *has totally geodesic fibres*
(3) $\mathbf{grad}e(\phi)$ *is horizontal.*

Proof. If the given conditions are satisfied then ϕ is a harmonic morphism from Corollary 5.2.

Now N^2 has *genus* $g \geq 1$, therefore it carries a hermitian metric of constant negative curvature or zero curvature. The proof then follows from [9, Theorem 2.5]. □

As an application we have

Corollary 5.4. *Let* M^{m+2} *be a compact Einstein manifold of positive scalar curvature and* N^2 *be a Riemann surface of genus* ≥ 1*. Then there exists no harmonic map* $\phi\colon M^{m+2}\to N^2$*, of rank 2 almost everywhere, such that* $\mathbf{grad}e(\phi)$ *is horizontal and* ϕ *has totally geodesic fibres.*

In contrast to Corollary 5.3, when N^2 is a compact Riemann surface of genus $g = 0$ the necessary determined conditions turn out to be sufficient as well.

Proposition 5.5. *Let* N^2 *be a compact Riemann surface of genus* $g = 0$ *and* M^{m+2} *be a closed Riemannian manifold. Let* $\phi\colon M^{m+2}\to N^2$ *be a non-constant submersive harmonic morphism with* $\mathbf{Ricci^M}(X_i, X_i) \geq K > 0$*,* $i = 1, 2$*, and dilation* $\lambda^2(x) \leq \frac{2K}{K^N}$ *where* $K^N > 0$ *denotes the curvature of* N^2 *and* (X_1, X_2) *is an orthonormal basis of* $T^H_x M$*. Then* ϕ *satisfies*

(1) $\mathbf{Ricci^M}(X_1, X_1) = \mathbf{Ricci^M}(X_2, X_2) = K > 0$*,*
(2) ϕ *has totally geodesic fibres,*
(3) $\mathbf{grad}e(\phi) = 0$*.*

Proof. Recall that the Weitzenböck formula for a harmonic morphism ϕ, subjected to the hypothesis, can be written as, cf. [9, Proposition 2.1]

$$\Delta\lambda^2 \leq -\|\nabla d\phi\|^2 + K^N\lambda^4 - 2K\lambda^2$$

which on integration implies that ϕ is totally geodesic, and as ϕ is non-constant we further have

$$\mathbf{Ricci^M}(X_1, X_1) = \mathbf{Ricci^M}(X_2, X_2) = K.$$

Moreover ϕ, being totally geodesic, essentially has totally geodesic fibres and constant dilation i.e satisfies $\mathbf{grad}e(\phi) = 0$. □

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