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LETTER TO THE EDITOR

A non-existence result for compact Einstein warped products

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Abstract

Warped products provide a rich class of physically significant geometric objects. The existence of compact Einstein warped products was questioned in Besse (1987 *Einstein Manifolds*, section 9.103). It is shown that there exists a metric on every compact manifold B such that (non-trivial) Einstein warped products, with base B, cannot be constructed.

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1. Introduction

Warped product construction is a construction in the class of Riemannian manifolds that generalizes direct product. This construction was introduced in [4] where it was used to construct a variety of complete Riemannian manifolds with negative sectional curvature. Warped products have significant applications, in general relativity, in the studies related to solutions of Einstein's equations [1, 2]. Besides general relativity, warped product structures have also generated interest in many areas of geometry, especially due to their role in construction of new examples with interesting curvature and symmetry properties cf [3, 5, 6, 9].

Definition 1.1. *Let (B, g_B) and (F, g_F) be Riemannian manifolds with $f : B \to (0, \infty)$ a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric*

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : B \times F \to B, \sigma : B \times F \to F$ are usual projections and $*$ denotes pullback. (B, g_B) is called the base, (F, g_F) is called the fibre and f is the warping function of the warped product.

If the warping function 'f' is constant then the warped product $B \times_f F$ (up to a change of scale) is a (global) Riemannian product, which we call as trivial warped product.

The reader is referred to [3, 9] for the fundamental results and properties of warped products.

A Riemannian manifold (M^m, g) is said to be Einstein if its Ricci curvature is a constant multiple of g. The notion of Einstein manifolds coincides with manifolds of constant curvature for $m \leqslant 3$, but Einstein manifolds constitute quite a large class in higher dimensions. Many new examples of Einstein manifolds have been obtained using warped products, cf [3]. Einstein warped products, due to their useful curvature and symmetry properties, provide a rich class of examples of practical interest in Riemannian as well as semi-Riemannian geometry. Yet there are no known examples of (non-trivial) compact Einstein warped products. This is what was questioned in [3, section 9.103]: *Can a (non-trivial) compact Einstein warped product be constructed?*

The purpose of this letter is to study this conjecture about non-existence of (non-trivial) compact Einstein warped products and to show that there exists a metric on every compact manifold B, such that a (non-trivial) Einstein warped product $M = B \times_f F$ cannot be constructed.

2. Main result

We begin by proving some necessary conditions for the existence of (non-trivial) Einstein warped products with compact base.

Proposition 2.1. *Let $M^m = B \times_f F$ be a warped product of an $(m-n)$-dimensional compact Riemannian manifold B and an n-dimensional Riemannian manifold F. Let Scal^B denote the scalar curvature of B. If M is Einstein with Einstein constant c^M and*

$$\text{either} \quad \mathrm{Scal}^B \leqslant (m-n)c^M \quad \text{or} \quad \mathrm{Scal}^B \geqslant (m-n)c^M$$

then the warping function f is constant and, up to a scale, M is a Riemannian product.

Proof. Using the well-known curvature identity [9, p 211] relating the Ricci curvatures of the warped product M and the base B, we have

$$-\frac{n}{f}\Delta^B f = \sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) - \mathrm{Scal}^B,$$

where Ric denotes the Ricci curvature of M and Δ^B is the Laplacian on B.

For the Einstein metric, the above equation becomes

$$-\frac{n}{f}\Delta^B f = (m-n)c^M - \mathrm{Scal}^B.$$

Now the conditions

$$\mathrm{Scal}^B \leqslant (m-n)c^M \quad \text{or} \quad \mathrm{Scal}^B \geqslant (m-n)c^M$$

make f a subharmonic or superharmonic function. Since B is compact, f must be constant. □

The above necessary conditions yield the following non-existence result for compact Einstein warped products.

Theorem 2.2. *There exists a metric on every compact manifold B such that there are no (non-trivial) Einstein warped products $M^m = B \times_f F$ with base B.*

Proof. For one-dimensional base with any metric, $\mathrm{Scal}^B \equiv 0$. Every two-dimensional manifold admits a metric of constant curvature. Every compact manifold of dimension at least 3 carries a metric of constant negative curvature [8]. Using proposition 2.1 and the above facts complete the proof. □

The question of existence of compact Einstein manifolds has also been addressed recently in [7], where it is shown necessarily for these manifolds to have positive scalar curvature.

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