Appendix II *Introduction to Matrices*

Part-3 Eigenvalues and eigenvectors

Here we study the concept of eigenvalues and eigenvectors. These are special scalars and vectors associated with matrices and have applications in variety of technical fields.

In Chapter 8 we will apply eigenvalues and eigenvectors to solve systems of linear differential equations. But these have a wide range of applications and you are likely to see these in your future engineering courses.

Learning Outcomes

After completing this sub-section, you will inshaAllah be able to

1. explain what is meant by eigenvalues and eigenvectors
2. find eigenvalues and eigenvectors
    - Real and Complex
3. find eigenvalues and eigenvectors using MATLAB

See class explanation

Remember to revise remarks about solving homogeneous linear systems of algebraic equations.
We will need the to understand this topic

A homogeneous system $A\mathbf{x} = 0$ has nontrivial solution iff $|A| = 0$

Introduction

- Given an $n \times n$ matrix A and an $n \times 1$ vector $\mathbf{v}$

  Question: What will be $A\mathbf{v}$?

  Ans: Another $n \times 1$ vector

- Example Take $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

  1. $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  2. $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Then $A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

  Special Example (3 and 4):

  3. $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then $A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\mathbf{v}$

  4. $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. $A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3\mathbf{v}$

- We see that generally the matrix A (on multiplication) transforms a vector $\mathbf{v}$ into another vector $A\mathbf{v}$.
- But for special vectors, the vector $A\mathbf{v}$ is just a scalar multiple of $\mathbf{v}$

> Such special vectors $\mathbf{v}$ and the corresponding scalar multiples are useful in study of many important problems (Engineering, Scientific)

> Hence our focus will be to study problems of the form $A\mathbf{v} = \lambda\mathbf{v}$
>
> where
> - A is a matrix
> - λ is a scalar
> - $\mathbf{v}$ is a vector

Definitions and The Eigenvalue Problem

Given an $n \times n$ matrix A.

A scalar λ is called an eigenvalue of matrix A if there is a non-zero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda \mathbf{v}.$$

The non-zero vector $\mathbf{v}$ is called eigenvector associated with eigenvalue λ.

- Need to find two unknowns λ and v from one equation
- See below the trick to handle this situation

Main idea for solving eigenvalue problems

- Given an $n \times n$ matrix A.
- To find eigenvalues & eigenvectors by solving $A\mathbf{v} = \lambda\mathbf{v}$ **(*)**
- **Stage 1:** To find all eigenvalues λ

To find λ for which Eq. (*) has non-zero solutions

We can rewrite (*) as

$$A\mathbf{v} - \lambda\mathbf{v} = 0$$

or $$A\mathbf{v} - \lambda I\mathbf{v} = 0$$

or $$(A - \lambda I)\mathbf{v} = 0 \quad (**)$$

To find λ for which Eq. (**) has non-zero solutions

Recall: A homogeneous system $A\mathbf{x} = 0$ has nontrivial solution iff $|A| = 0$.

- **For (**) to have non-trivial solution we must have**

$$|A - \lambda I| = 0 \qquad (1)$$

- **Solutions of Eq. (1) give the eigenvalues λ**

- **Stage 2:** To find eigenvectors

- **For each eigenvalue λ found in stage 1, solve Eq. (**) to find the corresponding eigenvectors**
- **i.e. solve the linear system $(A - \lambda I)\mathbf{v} = 0$ to find the corresponding non-trivial solutions**

Definitions of Characteristic Equation and Polynomial

- The Equation (1) on last page, i.e. $|A-\lambda I|=0$, is called the characteristics equation of matrix A
  - If A is an $n\times n$ matrix then $|A-\lambda I|$ is always a polynomial (of degree n) in powers of λ.
- The polynomial $p(\lambda)=|A-\lambda I|$ is called characteristic polynomial of matrix A.

Procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix A

- **Write the characteristic equation**

$$|A - \lambda I| = 0 \quad (*)$$

- **Find solutions of characteristic equation to get the eigenvalues** λ

- **For each eigenvalue λ_i, find the eigenvectors by solving the linear system**

$$(A - \lambda_i I)\mathbf{v} = 0$$

See Examples 1, 2, 3 done in class

Since the eigenvalues are just roots of a polynomial, it is possible to get repeated eigenvalues.

In the example 3 we will see eigenvectors associated to a repeated eigenvalue

Finding characteristic polynomial, eigenvalues and eigenvectors using MATLAB

First enter the $n \times n$ matrix A and then use the following sequence of commands

- **>> poly(A)**

Gives coefficients of characteristic polynomial

We can further use command **>> roots**. However we don't need it because of the following command

- **>> E= eig(A)**

Gives eigenvalues

- For each eigenvalue λ_i, solve $(A - \lambda_i I)\mathbf{v} = 0$ using

>> null($A - \lambda_i * eye(n)$**,'r')**

Gives independent eigenvectors (in each column)

See Examples 4, 5 done in class

MATLAB examples to be done in class

Example 4 [Distinct + Repeated Real Eigenvalues]

Use MATLAB to find the eigenvalues and associated eigenvectors of $\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Solution: Done in class.

Example 5 [Distinct + Repeated Real Eigenvalues]

Use MATLAB to find the eigenvalues and associated eigenvectors of

$$\begin{bmatrix} 15 & -6 & -18 & -6 \\ -4 & 5 & 8 & 4 \\ 12 & -6 & -15 & -6 \\ 4 & -2 & -8 & -1 \end{bmatrix}.$$

Solution: Hints in class.

Answer: Eigenvalues $\lambda_1 = -3, \lambda_2 = 3, \lambda_3 = 1, \lambda_4 = 3$

Eigenvectors:

$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda = -3$, $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ for $\lambda = 1$

and $\begin{bmatrix} 0.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ for repeated eigenvalue $\lambda = 3$.

Complex Eigenvalues

- Until now we have seen only real eigenvalues (which were roots of characteristic polynomial)
- Since the roots of characteristic polynomial can be imaginary numbers. Therefore, we can have complex eigenvalues.
- So we need to adapt our method for finding eigenvectors of complex eigenvalues (as shown in example below)

Example 6 [Complex Eigenvalues]

Find the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: Done in class

- we have conjugate pair of eigenvalues and the associated eigenvectors are also conjugate.

This is always the case.

- If $\lambda_1 = r$ is a complex eigenvalue then $\lambda_2 = \bar{r}$ is also an eigenvalue.
- The eigenvectors associated to $\lambda_2 = \bar{r}$ are just the conjugate of the eigenvectors associated with $\lambda_1 = r$.

End of Appendix Part 3
Do Qs. 47-56