Appendix II *Introduction to Matrices*

Part-2 Solving linear systems and finding matrix inverse using elementary row operations

Learning Outcomes

After completing this sub-section, you will inshaAllah be able to

1. explain what is meant by the row echelon form of a matrix
2. solve linear systems using Gaussian elimination method
3. explain what is meant by the reduced row echelon form of a matrix
4. solve linear systems using Gauss Jordan elimination method
5. find inverse of a matrix using elementary row operations

Elementary row operations

- All the methods learnt in this part depend heavily on use of operations called "Elementary Row Operations".
- These operations will be used to convert a matrix into nice & simpler form which will give solution of a linear system in a very simple manner.
- First we need to recall elementary row operations.

	Elementary row operations	Notation/example
1	Interchange two rows	R_{12}
2	Multiply any row by a non-zero constant	cR_1
3	Replace one row by "the sum of itself and a multiple of another row"	$R_2 + cR_1$

In order to solve a linear system, we will apply elementary row operations on augmented matrix of a linear system. Let's recall what is augmented matrix.

In general, the augmented matrix for a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}$$

is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{bmatrix}.$$

Gaussian elimination method

What is row echelon form of a matrix?
(Preparation for Gaussian elimination method)

A matrix is in echelon form if

1. All zero-rows (all elements zero) are below non-zero rows.
2. The first non-zero element of a row lies on the right of the first non-zero element of the previous row.
3. All elements below the first non-zero element of each row are zero.

Example: Which of the following are in echelon form?

(i) $\begin{bmatrix} -2 & 6 & 0 & 7 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 2 & 9 \\ 0 & 7 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 7 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 7 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

Terminology for matrices in echelon form:

- The first non-zero element of each row is called its leading entry.
- A column containing a leading entry is called a leading column
- In case of augmented matrices, the variables corresponding to leading columns are called leading variables and all others are called free variables.

Leading (or free) variables will play important role in writing solutions (see examples below).

How to convert a matrix into row echelon form?

(Preparation for Gaussian elimination method)

Use elementary row operations on the matrix in a systematic way as explained in the examples below. The process is called Gaussian elimination method.

Gaussian elimination method for converting a matrix into echelon form

- Begin with left most non-zero column (it's your first leading column)
- Choose the leading entry & bring at the top of current leading column (how? Of course by interchanging rows)
- Convert the entries below the leading entry into zeros.
- Jump to the next lower level
- Repeat the above steps until echelon form is achieved.

See Example 1 done in class

Calculation tips

- Finish the calculations of whole row and then go to next row.
- Avoid fractional calculations as much as possible (by interchanging rows).

Exercise:

Transform the matrix $\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$ into echelon form. Indicate the leading entries and corresponding leading columns.

Hint: you will need to change rows to avoid fractional calculations.

Solving linear systems using Gauss elimination method

Given a linear system. To find its solution (if it exists)

- Write the augmented matrix of the system
- Convert the augmented matrix to echelon form (using Gaussian elimination method)
- Write the corresponding linear system
- Decide about the leading and free variables (see Examples below)
- If the solution is possible, write the solution using backward substitution.

Example 2

Solve the system
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_1 + 8x_2 + 3x_3 &= 2 \\ 2x_1 + 7x_2 + 4x_3 &= 8 \end{aligned}.$$

Solution:

- The ***augmented matrix*** is $\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 8 & 3 & 2 \\ 2 & 7 & 4 & 8 \end{bmatrix}$
- Its ***echelon form*** is $\begin{bmatrix} \boxed{1} & 3 & 2 & 5 \\ 0 & \boxed{2} & -1 & -8 \\ 0 & 0 & \boxed{1/2} & 2 \end{bmatrix}$ **(*)**

  See example 1 for computations of echelon form

- The ***associated system*** is $\begin{aligned} x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_2 - x_3 &= -8 \\ \frac{1}{2}x_3 &= 2 \end{aligned}$ **(**)**
- x_1, x_2, x_3 are ***leading variables***. {there are ***no free variables***}

  Why

  Can U see the leading entries in (*)? Which are the leading columns in (*)

- From (**)
  - 3rd equation $\Rightarrow \boxed{x_3 = 4}$
  - Back substitution of x_3 in 2nd equation gives $\boxed{x_2 = -2}$
  - Back substitution of x_2, x_3 in 1st equation gives $\boxed{x_1 = 3}$

  Solution by back substitution

Unique solution: $x_1 = 3,\ x_2 = -2,\ x_3 = 4$

Example 3

Solve the system
$$\begin{aligned} x_1 \quad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 . \\ 3x_1 \quad -15x_3 &= 3 \end{aligned}$$

Solution:

- The ***augmented matrix*** is $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 3 & 0 & -15 & 3 \end{bmatrix}$

- Its ***echelon form*** is
$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (*)$$

See class notes for computations of echelon form

- The ***associated system*** is
$$\begin{aligned} x_1 \quad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 \quad (**) \\ 0 &= 0 \end{aligned}$$

- x_1, x_2 are ***leading variables***.
  x_3 is ***free variable***.

Why

Can U see the leading entries in (*)? Which are the leading columns in (*)

- From (**)
  - Choosing $\boxed{x_3 = t}$ (any arbitrary value)
  - From 2nd equation $x_2 = 4 - x_3$
    or $\boxed{x_2 = 4 - t}$
  - From 1st equation $x_1 = 1 + 5x_3$
    or $\boxed{x_1 = 1 + 5t}$

Solution by back substitution

- Infinite solutions: $x_1 = 1 + 5t$, $x_2 = 4 - t$, $x_3 = t$.

Remark: In backward substitution

- Set each free variable = an arbitrary parameter
- Solve the leading variables in terms of free variable or arbitrary parameter.

Gauss Jordan elimination method

What is reduced row echelon form of a matrix?

(Preparation for Gauss Jordan elimination method)

A matrix is in reduced echelon form

- if it is in echelon form with the following additional properties:
  1. Each leading entry is 1
  2. Each leading entry is the only non-zero entry in its column.

Example: Which of the following are in *reduced* echelon form?

(i) $\begin{bmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$

(iii) $\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 & 2 \\ 0 & \boxed{1} & 9 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 9 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 & 1 \\ 0 & \boxed{1} & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$

(v) $\begin{bmatrix} \boxed{1} & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$

(v) $\begin{bmatrix} \boxed{1} & 5 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$

Remember the terminology for matrices in echelon form:

- The first non-zero element of each row is called its leading entry.
- A column containing a leading entry is called a leading column
- In case of augmented matrices, the variables corresponding to leading columns are called leading variables and all others are called free variables.

Important fact

The reduced echelon form of a matrix is unique

How to convert a matrix into reduced row echelon form?

(Preparation for Gauss Jordan elimination method)

Use elementary row operations on the matrix in a systematic way as explained below. The process is called Gauss Jordan elimination method.

Gauss-Jordan elimination method for converting a matrix to *reduced* echelon form

- Convert the matrix into echelon form
- Divide every non-zero row by its leading entry
- Use each leading entry to convert the nonzero entries in its column, into zeros.

See Example 4 done in class

Solving linear systems using Gauss Jordan elimination method

Given a linear system. To find its solution (if it exists)

- Write the augmented matrix of the system
- Convert the augmented matrix to *reduced* echelon form (using Gauss-Jordan elimination method)
- Write the corresponding linear system
- Decide about the leading and free variables
- If the solution is possible, write the solution using backward substitution

Example 5

Solve $\begin{aligned} x_1 + 3x_2 - 2x_3 &= -7 \\ 4x_1 + x_2 + 3x_3 &= 5 \\ 2x_1 - 5x_2 + 7x_3 &= 19 \end{aligned}$

Solution

- **Augmented matrix** $\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & -11 & 11 & 33 \\ 2 & -11 & 11 & 33 \end{array}\right)$

- **Echelon form**

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & -11 & 11 & 33 \\ 2 & -11 & 11 & 33 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-4R_1} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right) \xrightarrow[R_3/-11]{R_2/-11} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \end{array}\right) \xrightarrow[R_3-R_2]{} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- **Reduced Echelon form**

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- **Associated system** implies $\left.\begin{aligned} x_1 &= -x_3 + 2 \\ x_2 &= x_3 - 3 \\ 0 &= 0 \end{aligned}\right\}$

x_3 is free variable

- Solution $x_3 = t$, $x_2 = t - 3$, $x_1 = -t + 2$

Finding matrix inverse using elementary row operations

Given a $n \times n$ matrix A. To find A^{-1}(if possible).

- Adjoin the identity matrix with A to form a matrix $[A \vdots I_n]$
- Compute the reduced echelon form of $[A \vdots I_n]$
  - If it is of the type $[I_n \vdots B]$ then B is inverse of A.
  - If it is ***not*** of the type $[I_n \vdots B]$ then A has ***no*** inverse.

See Example 6 done in class

End of Appendix Part 2
Do Qs. 31-45