Section 8.3 *Nonhomogeneous linear systems*

Variation of parameters

Learning outcomes

After completing this section, you will inshaAllah be able to

1. explain what is meant by general solution of a non-homogeneous linear system
2. write fundamental matrix of a homogeneous linear system
3. use variation of parameters to find particular solution of non-homogeneous linear systems
4. find general solution of non-homogeneous linear systems

General solution of non-homogeneous systems

- Given a non-homogeneous linear system

$$X' = AX + F \qquad (*)$$

- If X_c is the general solution of associated homogeneous system $X' = AX$

  and

  X_p is a particular solution of (*) then the general solution of (*) is given by

$$X = X_c + X_p$$

Complementary Solution

Particular Solution

- We know from 8.2, how to find the general solution X_c of $X' = AX$
- Here we learn how to find a particular solution X_p of (*)

Fundamental matrix of a homogeneous systems

- Given a homogeneous system $X' = AX$ with general solution

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$
$$= c_1\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + c_2\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

- We can write as $X = \begin{bmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

or $X = \phi(t) \cdot C$

$\phi(t)$: Fundamental Matrix of homogeneous system

Important property of fundamental matrix

If $\phi(t)$ is fundamental matrix of $X' = AX$ then

$$\phi'(t) = A\phi(t)$$

- $X = \phi(t) \cdot C \Rightarrow X' = \phi'(t) \cdot C$
- Using in $\Rightarrow X' = AX$ gives

$\phi'(t) = A\phi(t)$

Method of variation of parameters
(to find particular solution of non-homogeneous systems)

- Given a non-homogeneous linear system $X' = AX + F$ (*)

  and

  the general solution $X_c = \phi(t)\cdot C$ of associated homogeneous system $X' = AX$

Q: To use $X_c = \phi(t)\cdot C$ to find X_p

- Assume form of X_p as $X_p = \phi(t)\cdot U(t)$ (**)
- Putting $X_p = \phi(t)\cdot U(t)$ in (*) gives

$$\frac{d}{dt}\left(U(t)\right) = \phi^{-1}(t)\cdot F$$

See class discussion

- Hence $U(t) = \int \phi^{-1}(t)\cdot F(t)dt$
- Therefore, from (**) $X_p = \phi(t)\cdot \int \phi^{-1}(t)\cdot F(t)dt$

See example done in class

- We will need to find inverse $\phi^{-1}(t)$ of the matrix $\phi(t)$
- In general we can use the method learnt in Appendix.
- For the 2×2 matrices recall the following efficient formula

  If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

End of 8.3