Section 8.2 *Homogeneous linear systems with constant coefficients*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. see the relation of eigenvalues (& eigenvectors) of the matrix A with solutions of homogeneous linear system $X' = AX$
2. solve homogeneous systems $X' = AX$ if
   - A has distinct real eigenvalues
   - A has distinct complex eigenvalues
   - A has both distinct real and complex eigenvalues
3. solve homogeneous systems $X' = AX$ of n equations when
   - A has repeated eigenvalues (with n independent eigenvectors)
4. Use generalized eigenvectors to solve homogeneous systems $X' = AX$ of n equations when
   - A has repeated eigenvalues (with only one independent eigenvector)

Relation of eigenvalues (& eigenvectors) with solutions of homogeneous systems

- Consider a homogeneous linear system

$$X' = AX \qquad (*)$$

- Question: When a vector of the form $X = \mathbf{v}e^{\lambda t}$ will be a solution of (*)?
- Answer: $X = \mathbf{v}e^{\lambda t} \quad \Rightarrow \quad X' = \mathbf{v}\lambda e^{\lambda t}$

  Using X, X' in (*) we see that

  $$\mathbf{v}\lambda e^{\lambda t} = A\mathbf{v}e^{\lambda t} \quad \Leftrightarrow \quad \mathbf{v}\lambda = A\mathbf{v} \quad \text{or} \quad A\mathbf{v} = \lambda\mathbf{v}$$

$X = \mathbf{v}e^{\lambda t}$ is a solution of (*)

$$\Leftrightarrow$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

(i.e. λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector of A)

By knowing eigenvalues & eigenvectors we can find solutions of (*)

We will study solutions of homogeneous systems as following cases

- distinct real eigenvalues of A **{Part 1}**
- distinct complex eigenvalues of A **{Part 1}**
- repeated eigenvalues of A **{Part 2}**

Part 1

Distinct real or complex eigenvalues of A

Note

For a homogeneous system $X' = AX$ (with A an $n \times n$ matrix),

our aim will be to find n independent solutions

From last page

each eigenvector generates one solution

In case of distinct eigenvalues we have the following fact

If an $n \times n$ matrix has n distinct eigenvalues then we can always find n linearly independent eigenvectors.

Hence finding general solutions is simple in case of "Distinct real or complex eigenvalues of A"

Solving homogeneous linear systems

Distinct real eigenvalues

- Given $X' = AX$ (*) $(A:\ n\times n)$

  and A has n real distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

  - We need n independent solutions
  - Each eigenvalue will give one.

❖ For each eigenvalue λ_i, find an associated eigenvector $\mathbf{K}_i$

➢ The solution vector associated to this eigenvalue is $X_i = \mathbf{K}_i e^{\lambda_i t}$

❖ Find all solution vectors and write general solution as

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

or

$$X = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{K}_n e^{\lambda_n t}$$

See examples 1, 2 done in class

Solving homogeneous linear systems

Distinct complex eigenvalues

- Given $X' = AX$ (*)

  and A has distinct complex eigenvalues

- Recall: complex eigenvalues and eigenvectors occur in conjugate pairs.

❖ For each pair $\boxed{\alpha \pm i\beta}$ of eigenvalues, find the associated pair of eigenvectors and split into real and imaginary parts as $\boxed{\mathbf{B}_1 \pm i\mathbf{B}_2}$

➢ The solution vectors associated to the pair of eigenvalues $\alpha \pm i\beta$ are

$$\boxed{\begin{aligned} X_1 &= e^{\alpha t}\left(\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t\right) \\ X_2 &= e^{\alpha t}\left(\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t\right) \end{aligned}}$$

See Page-348 of the book for details of writing real form of solution

❖ Find all solution vectors and write general solution.

See example 3 done in class

Solving homogeneous linear systems

Mixture of previous two case of distinct eigenvalues

- Given $X' = AX$ (*)

  and A has distinct real and complex eigenvalues

- For each real eigenvalue λ_i, find eigenvector $\mathbf{K}_i$
  - Then $X_i = \mathbf{K}_i e^{\lambda_i t}$ is corresponding term in solution

- For each complex pair $\alpha \pm i\beta$ of eigenvalues, find the pair of eigenvectors $\mathbf{B}_1 \pm i\mathbf{B}_2$
  - Then

$$e^{\alpha t}\left(\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t\right)$$
$$e^{\alpha t}\left(\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t\right)$$

are corresponding terms in the general solution

See example 4 done in class

Part 2

Reperated eigenvalues of *A*

Note

An eigenvalue of multiplicity *k* can give

- *k* eigenvectors
- or less than *k* eigenvectors

If an eigenvalue of multiplicity k

- gives k independent eigenvectors then we have no problems and can write solution like Part 1
- gives less than k independent eigenvectors then we need special tricks

Hence we look at cases

- eigenvalue of multiplicity 2 giving 1 eigenvector
- eigenvalue of multiplicity 3 giving 1 eigenvector

For other cases, it is better to use matrix exponential, which we will study in section 8.4

Solving homogeneous linear systems {when repeated eigenvalues with multiplicity *k* have *k* independent eigenvetors}

- Given $X' = AX$
- If λ_i is an eigenvalue (of A) of multiplicity *k* with *k* independent eigenvectors, then
  - o find *k* independent eigenvectors

    $\mathbf{K}_1, \mathbf{K}_2, \cdots, \mathbf{K}_k$
  - o the solution terms associated to the eigenvalue λ_i are

$$c_1\mathbf{K}_1 e^{\lambda_i t} + c_2\mathbf{K}_2 e^{\lambda_i t} + \cdots + c_k\mathbf{K}_k e^{\lambda_i t}$$

See example 5 done in class

For the cases where

- eigenvalues of multiplicity 2 give 1 eigenvector

or

- eigenvalues of multiplicity 3 give 1 eigenvector

we will find generalized eigenvectors to get complete set of independent solutions of $X' = AX$.

First we learn how to find generalized eigenvectors of a matrix A

Generalized eigenvectors of a matrix A

- If λ is a repeated eigenvalue (of multiplicity 2) with one eigenvector $\mathbf{K}_1$ i.e. $(A-\lambda I)\mathbf{K}_1 = 0$
- Then a generalized eigenvector $\mathbf{K}_2$ for λ can be found by solving $(A-\lambda I)\mathbf{K}_2 = \mathbf{K}_1$

- If λ is a repeated eigenvalue (of multiplicity 3) with one eigenvector $\mathbf{K}_1$ i.e. $(A-\lambda I)\mathbf{K}_1 = 0$
- Then two generalized eigenvectors $\mathbf{K}_2$, $\mathbf{K}_3$ for λ can be found by solving $(A-\lambda I)\mathbf{K}_2 = \mathbf{K}_1$ and $(A-\lambda I)\mathbf{K}_3 = \mathbf{K}_2$

See example 6 done in class

Solving homogeneous linear systems
{when repeated eigenvalues of multiplicity *2* have *one* independent eigenvetor}

- Given $X' = AX$
- If λ_i is an eigenvalue (of A) of multiplicity 2 with *one* independent eigenvector $\mathbf{K}_1$,
  - find a generalized eigenvector $\mathbf{K}_2$ for λ_i by solving $(A - \lambda I)\mathbf{K}_2 = \mathbf{K}_1$
  - Then the solution terms associated to the eigenvalue λ_i are

$$c_1\mathbf{K}_1 e^{\lambda_i t} + c_2\left(\mathbf{K}_1 t + \mathbf{K}_2\right)e^{\lambda_i t}$$

See example 7 done in class

Solving homogeneous linear systems {when repeated eigenvalues of multiplicity *3* have *one* independent eigenvetor}

- Given $X' = AX$
- If λ_i is an eigenvalue (of A) of multiplicity *3* with *one* independent eigenvector $\mathbf{v}_1$,
    - o find two generalized eigenvectors $\mathbf{K}_2,\ \mathbf{K}_3$ for λ_i

      by solving $(A-\lambda I)\mathbf{K}_2 = \mathbf{K}_1$ and $(A-\lambda I)\mathbf{K}_3 = \mathbf{K}_2$
    - o Then the solution terms associated to the eigenvalue λ_i are

$$c_1\mathbf{K}_1 e^{\lambda_i t} + c_2\left(\mathbf{K}_1 t + \mathbf{K}_2\right)e^{\lambda_i t} + c_3\left(\mathbf{K}_1\frac{t^2}{2!} + \mathbf{K}_2 t + \mathbf{K}_3\right)e^{\lambda_i t}$$

See example 8 done in class

End of 8.2

Do Qs. 1-14, 19-31, 33-46.