Section 6.2 *Solutions about singular points*

We know from section 6.1, how to find series solutions about ordinary points. Here we extend that method to finding series solutions about regular singular points.

Learning Outcomes

After completing this sub-section, you will inshaAllah be able to

1. explain what are regular singular points
2. explain what is indicial equation and find indicial equation and its roots
3. use method of Frobenius to find series solution(s) of a 2nd order linear equations about regular singular points according to following different cases of roots r_1, r_2 of indicial equation
   a. $r_1 - r_2$ is not an integer
   b. $r_1 - r_2$ is an integer but $r_1 \neq r_2$
   c. $r_1 = r_2$
4. use reduction of order (learnt in 4.2) to find second solution using a known series solution (in situations where Frobenius method gives only one solution)

Preparation for Frobenius' method

Regular Singular Points

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).
- Given a singular point x_0.

> - **x_0 is regular singular point**
>
>   if
>
> $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$
>
>   are both analytic at x_0

Example: Is $x = 1$ a regular singular point of

$$(x-1)^2 y'' + 2x(x-1)y' + 3(x+1)y = 0 ?$$

Solution:

- Writing as $y'' + \dfrac{2x(x-1)}{(x-1)^2} y' + \dfrac{3(x+1)}{(x-1)^2} y = 0$, we see that
- $P(x) = \dfrac{2x}{x-1}$ and $Q(x) = \dfrac{3(x+1)}{(x-1)^2}$
- Hence $(x-1)P(x) = 2x$ is analytic at $x = 1$
- and $(x-1)^2 Q(x) = 3(x+1)$ is analytic at $x = 1$

Hence $x = 1$ is regular singular

Example: Is $x = 0$ a regular singular point of $x^2 y'' + 2(x-1)y' + xy = 0$?

Solution:

- Writing as $y'' + \dfrac{2(x-1)}{x^2} y' + \dfrac{1}{x} y = 0$, we see that
- $P(x) = \dfrac{2(x-1)}{x^2}$ and $Q(x) = \dfrac{1}{x}$
- Since $xP(x) = \dfrac{2(x-1)}{x}$ is not analytic at $x = 0$, it is not a regular singular point.

Ex: Find regular singular points of $\left(x^2 + x - 6\right) y'' + (x+3)y' + \left(x - 2\right) y = 0$.

Answer: $x = -3,\ x = 2$

What is the main aim of this section

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).
- If $x = x_0$ is a regular singular point then the aim is to learn to find solution(s) of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \quad \text{with } c_0 \neq 0 \qquad (**)$$

Here r is a number to be determined from an equation called indicial equation

So we first learn

- What is indicial equation and how to find it (if we try solutions of the form (**))

Note

- The roots of indicial equation will play important role in the method of this section

Preparation for Frobenius' method

How to find indicial equation?

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).
- And a regular singular point $x = x_0$.
- If we
  - Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ with $c_0 \neq 0$
  - Take necessary derivatives and put in (*)
  - Shift indices to write the equation as one series
- We get an equation in powers $(x - x_0)$

Separate the smallest power of $x - x_0$ in this equation

From our experience of previous section we know that to proceed to find the solution we will need to compare powers of $x - x_0$

Comparing coefficient of smallest power of x
gives an equation (of degree 2) in r
called the indicial equation

See Example 1 done in class

Finding indicial equation directly

- Given $y'' + P(x)y' + Q(x)y = 0$ and a regular singular point x_0.
- Set $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$.
- Then indicial equation is $r(r-1) + a_0 r + b_0 = 0$

  where $a_0 = p(0)$ and $b_0 = q(0)$.

Frobenius' Theorem

Assurance of at least one series solution about a regular singular point

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).
- If $x = x_0$ is an regular singular point then
  - we can always find one series solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

Converges in an interval around the point $x = x_0$

Here r_1 the larger root of indicial equation

Note

- Frobenius Theorem **guarantees at least one solution in all situations**.
- But in some situations we may get two solutions (see cases & examples below)

Our plan

- To learn to apply Frobenius method for different cases according to roots of indicial equation.
- So we carry out a case-by-case study below.

Method of Frobenius to find series solution near a regular singular point

Case 1: $r_1 - r_2 \neq$ positive integer

Case 1

The indicial equation has roots

r_1, r_2 with $r_1 > r_2$

and $r_1 - r_2 \neq$ positive integer

We have two solutions guaranteed.

One from each root $r = r_1,\ r = r_2$

of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n + r_1}$$

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n + r_2}$$

- **Below we find solutions for this case using Frobenius method**

Method of Frobenius to find series solution near a regular singular point
Case 1: $r_1 - r_2 \neq$ positive integer (contd)

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Choose the regular singular point to center the series solution.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$
- Take necessary derivatives and put in (*)
- Shift indices to write the equation as one series
- Compare coefficients of smallest power of x to get indicial equation
  - Find roots r_1, r_2 (with $r_1 \geq r_2$) of indicial equation. Check that $r_1 - r_2 \neq$ positive integer.
- Compare other coefficients and use the values of r_1, r_2 to get the corresponding recurrence relations.
- Use recurrence relations to find all c_n's for $r = r_1$ and $r = r_2$ separately.

  These give two independent solutions (one for each of r_1, r_2).
- Write the general solution.

See Example 2 done in class

Method of Frobenius to find series solution near a regular singular point
Case 2: $r_1 - r_2$=positive integer

Case 2

The indicial equation has roots

r_1, r_2 with $r_1 > r_2$

and $r_1 - r_2$=positive integer

We have one solution guaranteed from larger root $r = r_1$ of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

Our strategy

- We find one solution (which is guaranteed) by Frobenius method using the larger root.
- And then use this solution to find the other solution using reduction of order (learnt in 4.2)

- You may be tempted to try to find the second series solution using the second root $r = r_2$.
- Remember: there is no guarantee of second series solution for smaller root $r = r_2$. Sometimes it will give another independent solution and sometimes it will not.

- **Below we find solutions for this case.**
  - **One solution using Frobenius method**
  - **Second solution using reduction of order**

Method of Frobenius to find series solution near a regular singular point
Case 2: $r_1 - r_2$=positive integer (contd)

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Choose the regular singular point to center the series solution.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$
- Take necessary derivatives and put in (*)
- Shift indices to write the equation as one series
- Compare coefficients of smallest power of x to get indicial equation
  - Find roots r_1, r_2 (with $r_1 \geq r_2$) of indicial equation. Check that $r_1 - r_2 =$ positive integer.
- Compare other coefficients and use the **the larger root $r = r_1$** to get the recurrence relation.
- Use recurrence relations to find all c_n's for $r = r_1$.

  These give one solution.
- Find other solution by reduction of order.
- Write the general solution.

See Example 3 discussed in class

Recall Reduction of Order

- Given $y'' + P(x)y' + Q(x)y = 0$ and a solution $y_1(x)$
- Then a second solution $y_2(x)$ is given by $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$

Method of Frobenius to find series solution near a regular singular point

Case 3: $r_1 = r_2$

Case 3

The indicial equation has equal roots

i.e.

$r_1 = r_2$

We have one solution guaranteed for the root $r = r_1 = r_2$ of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

Our strategy

- We find one solution (which is guaranteed) by Frobenius method using the larger root.
- And then use this solution to find the other solution using reduction of order (learnt in 4.2)

- **Below we find solutions for this case.**
  - **One solution using Frobenius method**
  - **Second solution using reduction of order**

Method of Frobenius to find series solution near a regular singular point
Case 3: $r_1 = r_2$ (contd)

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Choose the regular singular point to center the series solution.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$
- Take necessary derivatives and put in (*)
- Shift indices to write the equation as one series
- Compare coefficients of smallest power of x to get indicial equation
  - Find roots r_1, r_2 of indicial equation. Check that $r_1 = r_2$.
- Compare other coefficients and use the root $r = r_1 = r_2$ to get the recurrence relation.
- Use recurrence relations to find all c_n's.

These give one solution.

- Find other solution by reduction of order.
- Write the general solution.

Note tricks to handle series in order to find y_2

See Example 4 discussed in class

Recall Reduction of Order

- Given $y'' + P(x)y' + Q(x)y = 0$ and a solution $y_1(x)$
- Then a second solution $y_2(x)$ is given by $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$

Remark-1 about solutions by Frobenius' method

Lottery situation in Case 2: $r_1 - r_2$=positive integer

- From above we have guarantee of one solution for the larger root. That is why we used the larger root to find first solution in above examples.
- Sometimes, the smaller root <u>may</u> give two independent solutions as we see in the example below.

Example 5

Consider

$$x(x-1)y'' + 3y' - 2y = 0$$

and assume solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ about the regular singular point $x_0 = 0$.

a) Show that the indicial equation has roots $r_1 = 4,\ r_2 = 0$.

We know by Frobenius theorem that the larger root will surely give one series solution

b) Use smaller root $r_2 = 0$ in Frobenius method to find two independent solutions.

Solution discussed in class

Remark-2 about solutions by Frobenius' method

Knowing the number of expected series solutions by directly writing the indicial equation (i.e. without solving the ODE)

- This remark will be explained through example and class discussion.

See Examples 6 done in class

End of 6.2