# Section 6.1 *Solution about ordinary points*

## 6.1.2 Power series solutions

Here we learn to find power series solutions to linear differential equations near ordinary points (see details below). We will stick to 2$^{nd}$ order equations but the method extends easily to higher order equations.

### Learning Outcomes

After completing this sub-section, you will inshaAllah be able to

1. explain the difference between ordinary and singular points
2. find series solution of a 2$^{nd}$ order linear equations near an ordinary point

# Preparation for power series solution method

## Ordinary and Singular Points

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).

- **Ordinary points:**
  Where both $P(x)$ and $Q(x)$ are analytic
- **Singular points:**
  Which are not ordinary points

**Example:** Find all ordinary points of $xy'' + \frac{x}{1-x}y' + (\sin x)\,y = 0$

**Solution:**

- Writing as $y'' + \frac{1}{1-x}y' + \left(\frac{\sin x}{x}\right)y = 0$
- $P(x) = \frac{1}{1-x}$ is not analytic only at $x = 1$
- $Q(x) = \frac{\sin x}{x}$ is analytic everywhere.

Because

$$Q(x) = \frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{3!} - \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

- In our course we will mostly be dealing with cases where $P(x)$ and $Q(x)$ are rational functions.
- We know from $6.1.1_5$ how to check if they are analytic or not.

- Hence only singular point is $x = 1$ and all other points are ordinary points.

## Preparation for power series solution method

## Existence of power series solution

- Given $y'' + P(x)y' + Q(x)y = 0$ (*).
- If $x = x_0$ is an ordinary point then
  - we can always find power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Converges in an interval around the point $x = x_0$

Called solution about the ordinary point $x = x_0$

The minimum radius of convergence is the distance of $x = x_0$ from the closest singular point.

## Next we learn the method of finding series solution

### When can we use this method?

- To solve linear homogeneous equations near an ordinary point
  - We will only consider 2nd order but it works for higher order equations

### What is the main idea of the method?

- Given $y'' + P(x)y' + Q(x)y = 0$ and ordinary point $x_0$.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
- Take derivatives and put in ODE
- Write as single series and get recurrence relation
- Use recurrence relation to find coefficients $c_n$

See below for details

## Method of finding series solution near an ordinary point

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Choose the ordinary point to center the series. If 0 is available, it often works nicely.
- Consider the solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
- Take necessary derivatives and put in (*)
- Shift indices to write the equation as one series
- Compare coefficients and get the recurrence relation.
- Use recurrence relation to find all $c_n$'s in terms of $c_0$ and $c_1$.

  These give two independent solutions.
- Write general solution in the form of series or write first few terms.

See Examples 1, 2, 3, 4 done in class

*End of 6.1.2*
*Do Qs. 15-34*