

Exercise 6.1.2

Q 19 Find series solution of  $y'' - 2xy' + y = 0$  about the ordinary point  $x = 0$ .

Solution:

• **Step1:** Let  $y = \sum_{n=0}^{\infty} c_n x^n$  be solution of the ODE.

• **Step2**  $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$ , and  $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$

• **Step2** The ODE becomes:

$$\begin{aligned} & \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - 2 \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k - 2 \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=0}^{\infty} c_k x^k \\ &= \underbrace{2c_2 + \sum_{k=1}^{\infty} c_{k+2} (k+2)(k+1) x^k}_{\text{}} - 2 \sum_{k=1}^{\infty} c_k k x^k + \underbrace{c_0 + \sum_{k=1}^{\infty} c_k x^k}_{\text{}} \\ &= \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_k(2k-1)] x^k + 2c_2 + c_0 = 0 \end{aligned}$$

• **Step3** Compare coefficients to get:

- $c_2 = -c_0 / 2$
- $c_{k+2} = \frac{(2k-1)c_k}{(k+1)(k+2)} \forall k = 1, 2, 3 \dots$

• **Step4** Evaluation of coefficients from recursive relation

$$c_2 = -c_0 / 2$$

$$c_3 = \frac{c_1}{(2)(3)}$$

$$c_4 = \frac{3c_2}{(3)(4)} = -\frac{c_0}{2.4} = -\frac{c_0}{8}$$

$$c_5 = \frac{(5)c_3}{(4)(5)} = \frac{c_3}{4} = \frac{c_1}{2.3.4}$$

$$c_6 = \frac{(7)c_4}{(5)(6)} = -\frac{7}{5.6.8}c_0 \text{ etc}$$

- **Step5** Plug the above values in the polynomial and writing solutions:

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$$

$$y = c_0 + c_1x - \frac{c_0}{2}x^2 + \frac{c_1}{6}x^3 - \frac{c_0}{8}x^4 + \frac{c_1}{24}x^5 - \frac{7c_0}{240}x^6 + \dots$$

$$y = c_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \dots\right) + c_1 \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \dots\right)$$

$$y = c_0y_0 + c_1y_1$$

where

$$y_0 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \dots = 1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 + \frac{3.7}{6!}x^6 + \dots$$

and

$$y_1 = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \dots = x + \frac{3}{3!}x^3 + \frac{5}{5!}x^5 + \dots$$

- **Step6** **Particular Solutions**

When  $c_1 = 0$  we get one particular solution  $y_0$

When  $c_0 = 0$  we get one particular solution  $y_1$

Q 27 Find series solution of  $(x^2 + 2)y'' + 3xy' - y = 0$  about the ordinary point  $x = 0$ .

Solution:

• **Step1:** Let  $y = \sum_{n=0}^{\infty} c_n x^n$  be solution of the ODE.

• **Step2**  $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$ , and  $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

• **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^n + 2 \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 3 \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

• **Step3** Change " $n - 2$ " to " $k$ " in the middle term so that we can take " $x^n$ " common

$$\sum_{k=2}^{\infty} c_k k(k-1)x^k + 2 \underbrace{\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k}_{k=0} + 3 \underbrace{\sum_{k=1}^{\infty} c_k k x^k}_{k=1} - \underbrace{\sum_{k=0}^{\infty} c_k x^k}_{k=0} = 0$$

• **Step4** Open summation over " $k = 0$ " above as below

$$\sum_{k=2}^{\infty} c_k k(k-1)x^k + 4c_2 + 12c_3x + 2 \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k +$$

$$+ 3c_1x + 3 \sum_{k=2}^{\infty} c_k k x^k - c_0 - c_1x - \sum_{k=2}^{\infty} c_k x^k = 0$$

• **Step5** Simplify above expression:

$$\sum_{k=2}^{\infty} [c_k k(k-1) + 2c_{k+2}(k+1)(k+2) + 3c_k k - c_k] x^k +$$

$$4c_2 + 12c_3x + 2c_1x - c_0 = 0$$

- **Step6** Compare coefficients of " $x^k$ ", " $x$ " and constant term to get:

$$c_k(k^2 + 2k - 1) + 2c_{k+2}(k+1)(k+2) = 0$$

$$\Rightarrow c_{k+2} = -\frac{k^2 + 2k - 1}{2(k+1)(k+2)}c_k, k = 2, 3, \dots$$

$$4c_2 - c_0 = 0$$

$$6c_3 + c_1 = 0$$

- **Step7** From the last two equations we get:

$$c_0 = 4c_2 \Rightarrow c_2 = c_0 / 4$$

$$c_1 = -6c_3 \Rightarrow c_3 = -c_1 / 6$$

- **Step8** Use recursive relation  $x$  to determine solutions as follows:

$$\bullet k = 2 \Rightarrow c_4 = -\frac{7}{24}c_2 = \frac{-7}{96}c_0 = \frac{-7}{4 \cdot 4!}$$

$$\bullet k = 3 \Rightarrow c_5 = -\frac{7}{20}c_3 = \frac{7}{120}c_1 = \frac{7}{5!}c_1$$

$$\bullet k = 4 \Rightarrow c_6 = -\frac{23}{60}c_4 = \frac{-23}{240}c_0 = \frac{-23}{2 \cdot 5!}c_0$$

- **Step9** Writing solution

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y = c_0 + c_1x + \frac{c_0}{4}x^2 - \frac{c_1}{6}x^3 - \frac{7c_0}{96}x^4 + \frac{7c_1}{120}x^5 + \frac{-23c_0}{240}x^6 + \dots$$

$$y = c_0(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 - \frac{23}{240}x^6 + \dots) + c_1(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \dots)$$

$$y = c_0y_1 + c_1y_1$$

- **Step10** **Particular Solutions**

When  $c_1 = 0$  we get one particular solution  $y_0$

When  $c_0 = 0$  we get one particular solution  $y_1$

Q 31 Use power series method to solve  $y'' - 2xy' + 8y = 0$  when  $y(0) = 0, y'(0) = 0$

Solution:

• **Step1:** Let  $y = \sum_{n=0}^{\infty} c_n x^n$  be series solution of the given ODE.

• **Step2**  $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$ , and  $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

• **Step2** The ODE becomes:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2 \sum_{n=1}^{\infty} c_n n x^n + 8 \sum_{n=0}^{\infty} c_n x^n = 0$$

• **Step3** Do the following in above expression:

• In first term:  $n - 2 = k \Rightarrow n = k + 2$

• In 2<sup>nd</sup> term:  $n = k$

• In 3<sup>rd</sup> term:  $n = k$

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1)x^k - 2 \sum_{k=1}^{\infty} c_k k x^k + 8 \sum_{k=0}^{\infty} c_k x^k = 0$$

• **Step4** Open summation in **first** and **third** term up to  $k=0$  as below:

$$2c_2 + \sum_{k=1}^{\infty} c_{k+2} (k+2)(k+1)x^k - 2 \sum_{k=1}^{\infty} c_k k x^k + 8c_0 + 8 \sum_{k=1}^{\infty} c_k x^k = 0$$

• **Step5** Re-write above as:

$$2c_2 + 8c_0 + \sum_{k=1}^{\infty} (c_{k+2} (k+2)(k+1) - 2c_k k + 8c_k) x^k = 0$$

• **Step6** Above gives:

$$c_{k+2} = \frac{2(k-4)}{(k+1)(k+2)} c_k \quad \forall k = 1, 2, 3, \dots \text{ and}$$

$$c_2 = -4c_0$$

- **Step7** From recursive relation write coefficients as:

- $k = 1 \Rightarrow c_3 = \frac{-6}{6}c_1 = -c_1$
- $k = 2 \Rightarrow c_4 = \frac{-4}{12}c_2 = -\frac{1}{3}(-4c_0) = \frac{4}{3}c_0$
- $k = 3 \Rightarrow c_5 = \frac{-2}{20}c_3 = -\frac{1}{10}(-c_1) = \frac{1}{10}c_1$
- $k = 4 \Rightarrow c_6 = 0$

Since all of  $c_8 = 0, c_{10}$  etc will be expressed in terms of each other, they are all zero. We therefore evaluate only the odd terms only.

- $k = 5 \Rightarrow c_7 = \frac{2}{42}c_5 = \frac{1}{21}\left(\frac{1}{10}c_1\right) = \frac{1}{210}c_1$  etc

- **Step8** Writing solution

- $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$
- $y = c_0 + c_1x - 4c_0x^2 - c_1x^3 + \frac{4}{3}c_0x^4 + \frac{1}{10}c_1x^5 + \frac{1}{210}c_1x^7 + \dots$
- $y = c_0\left(1 - 4x^2 + \frac{4}{3}x^4\right) + c_1\left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \dots\right)$

- **Step9** Use initial conditions

- $y(0) = 3 \Rightarrow c_0 = 3$
- $y'(0) = 0 \Rightarrow c_1 = 0$

- **Step10** The solution of the IVP is:

- $y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right)$

Q 34 Use power series method to solve  $y'' + e^x y' - y = 0$  about ordinary point  $x_0 = 0$

Solution:

• **Step1:** Let  $y = \sum_{n=0}^{\infty} c_n x^n$  be series solution of the given ODE.

• **Step2**  $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$ , and  $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$

• **Step3** Write series expression of  $e^x$  about  $x_0 = 0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

• **Step4** Substitute above expressions in the ODE to write it as:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots) \sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

• **Step5** Expand the above expression as:

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots) + (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots)(c_1 + 2c_2x + 3c_3x^2 + \dots) - (c_0 + c_1x + c_2x^2 + c_3x^3 + \dots) = 0$$

• **Step6** Re-write above as:

$$(2c_2 + c_1 - c_0) + (6c_3 + 2c_2 + c_1 - c_1)x + (12c_4 + 3c_3 + 2c_2 + \frac{c_1}{2})x^2 + \dots = 0 \bullet$$

**Step7** Compare and get:

$$2c_2 + c_1 - c_0 = 0$$

$$\Rightarrow c_2 = (c_0 - c_1)/2$$

$$3c_3 + c_2 = 0$$

$$\Rightarrow c_3 = -c_2/2 = (c_0 - c_1)/4$$

$$12c_4 + 3c_3 + 2c_2 + \frac{c_1}{2} = 0$$

$$\Rightarrow c_4 = -(c_1/2 + 2c_2 + 3c_3)/12$$

$$\Rightarrow c_4 = -\left[\frac{c_1}{2} + c_0 - c_1 - \frac{c_0}{2} + \frac{c_1}{2}\right]/12$$

$$\Rightarrow c_4 = -\left[\frac{c_0}{2} - c_1\right] = c_1 - c_0/2$$

**Step8** Substitute above values in

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

and re-arrange terms to get the answer.