Example 3 [Section 6.2]

Find a series solution y_1 of

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \qquad (*)$$

about $x = 0$. Use reduction of order to find the second solution.

Solution

- **Choosing the regular singular point x_0**

  - Clearly $x_0 = 0$ is a regular singular point.

- **Considering series solution y and putting y, y', y'' in (*)**

  - Take $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.
  - Then $y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$.

- **Putting in (*) & Simplifying**

  - Putting in (*) gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \frac{1}{4}\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \quad x^r \left\{ \sum_{n=0}^{\infty} \left[(n+r)(n+r-1)c_n + (n+r)c_n - \frac{1}{4}c_n \right] x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \right\}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \left(n+r-\frac{1}{2}\right)\left(n+r+\frac{1}{2}\right) c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2}$$

- **Shifting index**

  - We have the equation

$$\sum_{n=0}^{\infty} \left(n+r-\frac{1}{2}\right)\left(n+r+\frac{1}{2}\right) c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2}$$

Put $k = n$ (first sum)

Put $k = n+2$ (second sum)

$$\Rightarrow \sum_{k=0}^{\infty} \left(k+r-\frac{1}{2}\right)\left(k+r+\frac{1}{2}\right) c_k x^k + \sum_{k=2}^{\infty} c_{k-2} x^k$$

• Writing as single series

- Above equation implies

$$\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)c_0+\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)c_1x^1+\sum_{k=2}^{\infty}\left[\left(k+r-\frac{1}{2}\right)\left(k+r+\frac{1}{2}\right)c_k+c_{k-2}\right]x^k=0$$

for $k=0$ for $k=1$

• Comparing coefficients of powers of x

Obtained by comparing coefficient of smallest power

- Comparing coefficients of powers of x gives

$$\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)=0$$

• Indicial equation

$r=\frac{1}{2}$ larger root

$$\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)c_1=0$$

$$\left(k+r-\frac{1}{2}\right)\left(k+r+\frac{1}{2}\right)c_k+c_{k-2}=0 \qquad \text{for } k\ge 2$$

• Writing recurrence and other relations for chosen value of r

- For $r=\frac{1}{2}$, we get from above

$c_1=0$

$$c_k=-\frac{c_{k-2}}{k(k+1)} \qquad \text{for } k\ge 2$$

Recurrence relation to determine coefficients c_n

• Finding coefficients c_n's

- $k=2 \Rightarrow \quad c_2=-\frac{c_0}{2\cdot 3}=-\frac{c_0}{3!}$
- $k=3 \Rightarrow \quad c_3=-\frac{c_1}{3\cdot 4}=0$
- $k=4 \Rightarrow \quad c_4=-\frac{c_2}{4\cdot 5}=\frac{c_0}{5!}$
-
- $\vdots$

• Writing the solution y_1

- $y_1 = \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}}$ implies

$$y_1 = x^{1/2}\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\cdots\right)$$

$$= c_0 x^{1/2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\cdots\right)$$

Using above

$$= c_0 x^{1/2} \cdot \frac{x^{1/2}}{x^{1/2}}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\cdots\right)$$

May not be possible always

$$= \frac{c_0}{x^{1/2}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right)$$

$$= \frac{c_0}{x^{1/2}} \sin x$$

Writing solution in a better form

• Finding second solution using reduction of order

- Recall the formula $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$

- We have $P(x) = \frac{1}{x}$ and $y_1 = \frac{\sin x}{x^{1/2}}$. This immediately gives

$$y_2 = y_1 \int \csc^2 x dx = -\frac{\cos x}{x^{1/2}}$$

• Writing general solution

- Hence the general solution is

$$y = C_1 \frac{\sin x}{x^{1/2}} + C_2 \frac{\cos x}{x^{1/2}}$$

Some Remarks

- Though in above example we got a nice form of y_1
- In general, y_1 will be in the form of series.
- So we will have to use tricks for handling series in order to find y_2 (like example 4 done in class).

Do Q. 28 for further understanding of this point

Even if we find y_1 as a nice elementary function, the other solution y_2 may still be in the form of series because of complicated integration

For example in Q-29,

- the first solution is $y_1 = e^x$
- but the reduction of order formula for second solution gives

$$y_2 = y_1 \int \frac{e^{-x}}{x} dx .$$

  - For the integration we use the series expansion of e^{-x} and integrate after writing in the form

$$y_2 = y_1 \int \frac{1}{x} \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right] dx$$

Do Q. 29 for further understanding of this point