

Example 3 [Section 6.1.2]

Solve the IVP

$$(x-1)y'' - xy' + y = 0 \quad (*)$$

$$y(0) = -2, \quad y'(0) = 6$$

Solution

• Choosing ordinary point x_0

- We choose $x_0 = 0$ [since initial condition is at $x = 0$]

• Considering series solution y and putting y, y', y'' in (*)

- Take $y = \sum_{n=0}^{\infty} c_n x^n$.
- Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.
- Putting in (*) gives

$$(x-1) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

• Simplifying & shifting index

- Above equation can be written as

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

Put $k = n - 1$

Put $k = n - 2$

Put $k = n$

Put $k = n$

$$\Rightarrow \sum_{k=1}^{\infty} (k+1)k c_{k+1} x^k - \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

Writing each series in the form involving x^k

• Writing as single series

- Above equation implies

$$-2c_2 + c_0 + \sum_{k=1}^{\infty} [-(k+2)(k+1)c_{k+2} + (k+1)k c_{k+1} - (k-1)c_k] x^k = 0$$

• Comparing coefficients of powers of x

- Comparing coefficients of powers of x gives

$$-2c_2 + c_0 = 0$$

$$-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k = 0 \quad \text{for } k \geq 1$$

- Which gives

$$c_2 = \frac{c_0}{2}$$

$$c_{k+2} = \frac{k}{k+2}c_{k+1} - \frac{(k-1)}{(k+2)(k+1)}c_k$$

Recurrence relation to determine coefficients c_n

for $k \geq 1$

• Finding coefficients c_n 's

- $k=1 \Rightarrow c_3 = \frac{1}{3}c_2 - 0c_1 = \frac{c_0}{6}$

- $k=2 \Rightarrow c_4 = \frac{2}{4}c_3 - \frac{1}{4 \cdot 3}c_1 = \frac{1}{2} \cdot \frac{c_0}{6} - \frac{1}{12} \cdot \frac{c_0}{2} = \frac{c_0}{24}$

- \vdots

• Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$= c_1x + c_0 + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \dots$$

Using above

$$= c_1x + c_0 \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right)$$

General solution

• Writing solution of IVP

- From above we have $y' = c_1 + c_0 \left(\frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \dots \right)$

- Hence $y(0) = -2 \Rightarrow c_0 = -2$

and $y'(0) = 6 \Rightarrow c_1 = 6$

- Therefore solution of IVP is

$$y = 6x - 2 \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right).$$

- Writing solution in better form

May not be possible always

- We have found the solution

$$y = 6x - 2 \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right)$$

Is this a well known power series of some function?

- We can write as

$$y = 6x - 2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) + 2x$$

- Which implies

$$y = 8x - 2(e^x)$$

- Hence the solution of IVP is

$$y = 8x - 2e^x$$

Example 4 [Section 6.1.2]

Solve $y'' + \sin(x)y = 0$ (*)
about the ordinary point $x_0 = 0$.

Solution

- Considering series solution y and putting y, y', y'' in (*)

- Take $y = \sum_{n=0}^{\infty} c_n x^n$.
- Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.
- Putting in (*) gives

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + (\sin x) \sum_{n=0}^{\infty} c_n x^n = 0$$

To proceed we need to write $\sin x$ as a power series

- Putting series expansion of $\sin x$ and simplifying

- Using series expansion of $\sin x$ we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \left(x - \frac{x^3}{3!} + \dots \right) \sum_{n=0}^{\infty} c_n x^n = 0$$

- This implies

$$\begin{aligned} & [2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots] + \\ & + \left(x - \frac{x^3}{3!} + \dots \right) [c_0 + c_1x + c_2x^2 + \dots] = 0 \end{aligned}$$

How to handle such products?

See below.

$$\begin{aligned} \left(x - \frac{x^3}{3!} + \dots \right) [c_0 + c_1x + c_2x^2 + \dots] &= (c_0x + c_1x^2 + c_2x^3 \dots) + \left(-c_0 \frac{x^3}{3!} - c_1 \frac{x^4}{3!} - c_2 \frac{x^5}{3!} \dots \right) + \dots \\ &= c_0x + c_1x^2 + \left(c_2 - \frac{c_0}{6} \right) x^3 + \dots \end{aligned}$$

- Hence series becomes

$$\begin{aligned} & [2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots] + \left[c_0x + c_1x^2 + \left(c_2 - \frac{c_0}{6} \right) x^3 + \dots \right] = 0 \\ \Rightarrow & 2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + \left(20c_5 + c_2 - \frac{c_0}{6} \right) x^3 + \dots = 0 \quad (**) \end{aligned}$$

- Comparing coefficients of powers of x to get c_n 's

- Comparing coefficients of powers of x in (**) gives

$$c_2 = 0$$

$$c_3 = -\frac{c_0}{6}$$

$$c_4 = -\frac{c_1}{12}$$

$$c_5 = -\frac{c_2}{20} + \frac{c_0}{120} = \frac{c_0}{120}$$

⋮

- Writing general solution

- $y = \sum_{n=0}^{\infty} c_n x^n$ implies

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= \left(c_0 - c_0 \frac{x^3}{6} + c_0 \frac{x^5}{120} + \dots \right) + \left(c_1 x - c_1 \frac{x^4}{12} + \dots \right)$$

$$= c_0 \left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \dots \right) + c_1 \left(x - \frac{x^4}{12} + \dots \right)$$

General solution