Section 4.7 *Cauchy Euler equation*

Here we will study a class of linear equations with variable coefficients.

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. recognize a Cauchy Euler equation
2. solve a Cauchy Euler equation by two methods
   a. Method-1

      Using the substitution $y = x^m$

   b. Method-2

      By converting the Cauchy Euler equation into a linear equation with constant coefficients

Cauchy Euler equation

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x) \qquad (*)$$

Examples

- $3x^2 y'' - xy' + y = 0$
- $3x^2 y'' + y = 0$
- $3x^2 y'' - xy' = e^x$

- Because the coefficient of $\frac{d^n y}{dx^n}$ is zero at $x = 0$, we focus on studying the solution of ODE (*) on the interval $(0, \infty)$.
- Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ in ODE (*)

Method 1

Solving

Cauchy Euler differential equation

using

the substitution $y = x^m$

We first <u>focus on second order</u>

and

then consider equations of higher order

Main idea for solving second order Cauchy Euler equation using substitution $y = x^m$

1 Given a second order Cauchy Euler equation

$$ax^2y'' + bxy' + cy = 0 \qquad (*)$$

2 Get the auxiliary equation.

Assume solution of the form $y = x^m$.

$\Rightarrow \quad y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$.

Putting in (*)

$\Rightarrow \quad ax^2m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0$

$\Rightarrow \quad \left[am(m-1) + bm + c\right]x^m = 0$

$\Rightarrow \quad am^2 + (b-a)m + c = 0 \qquad (**)$

Equation (**) is called **Auxiliary Equation**

Note

$y = x^m$ is a solution of (*)

$\Leftrightarrow$

m is a root of (**)

2 Find the roots of auxiliary equation.

Since the auxiliary equation is a quadratic equation,

we have three possibilities:

- **2 distinct real roots**
- **1 real (repeated) root**
- **2 complex conjugate roots**

3 Write the two linearly independent solutions according to the type of the roots found in Step 2.

On next three pages, we see how to write independent solutions for different cases of roots

Case 1:

Solving second order Cauchy Euler equation

when auxiliary equation has 2 distinct real roots

- Given $ax^2y'' + bxy' + cy = 0$ (*)

- Assume solution $y = x^m$ to get the **auxiliary equation**

$$am^2 + (b-a)m + c = 0 \quad (**)$$

- Find **roots** of auxiliary equation.

  If they are 2 distinct real roots $m = m_1, m = m_2$

- Then $y_1 = x^{m_1},\ y_2 = x^{m_2}$ **are linearly independent solutions**

  of (*) and the **general solution** is given by $y = c_1 x^{m_1} + c_2 x^{m_2}$.

See Example 1 done in class

Case 2:

Solving second order Cauchy Euler equation when auxiliary equation has repeated real roots

- Given $ax^2y'' + bxy' + cy = 0$ **(*)**

- Assume solution $y = x^m$ to get the **auxiliary equation**

$$am^2 + (b-a)m + c = 0 \qquad \textbf{(**)}$$

- Find **roots** of auxiliary equation.

  If there is 1 real root $m = m_1$

  - Second solution found using 4.2
  - See class explanation

- Then $y_1 = x^{m_1},\ y_2 = x^{m_1}\ln x$ **are linearly independent solutions** of (*) and the **general solution** is given by $y = c_1x^{m_1} + c_2x^{m_1}\ln x$.

See Example 2 done in class

Case 3:
Solving second order Cauchy Euler equation when auxiliary equation has complex (conjugate) roots

- Given $ax^2y'' + bxy' + cy = 0$ (*)

- Assume solution $y = x^m$ to get the **auxiliary equation**

$$am^2 + (b-a)m + c = 0 \qquad (**)$$

- Find **roots** of auxiliary equation.

  If there are 2 complex (conjugate) roots $m = \alpha \pm i\beta$

See class explanation

- Then $y_1 = x^\alpha \cos(\beta \ln x),\ y_2 = x^\alpha \sin(\beta \ln x)$ **are linearly independent solutions** of (*) and the **general solution** is given by $y = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x)$.

See Example 3 done in class

Recall the remark on first page

- Because the coefficient of y'' in

$$ax^2y'' + bxy' + cy = 0 \qquad (*)$$

is zero at $x = 0$, we focus on studying the solution of ODE (*) on the interval $(0, \infty)$.

- Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ in ODE (*)

See Example 4 done in class

Solving higher order Cauchy Euler equations using substitution $y = x^m$

1 **Given a Cauchy Euler equation**

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0 \qquad (*)$$

2 **Assume solution of the form $y = x^m$ and get the auxiliary equation.**

3 **Find the roots of auxiliary equation.**

4 **For each root, write the corresponding term(s) of the general solution of (*)**

See next page, to learn how to write solution for different kinds of roots

5 **Write the general solution**

Solving higher order Cauchy Euler equations

Form of the solution for each kind of root

- **Distinct real roots**
  For each distinct real root $m = m_1$, there will be a term of the form cx^{m_1} in the general solution.
- **Repeated real roots**
  For each repeated real root $m = m_1$ of multiplicity k,
  there will be k terms of the form
  $c_1 x^{m_1},\ c_2 x^{m_1} \ln x,\ c_3 x^{m_1} (\ln x)^2, \cdots\cdots, c_k x^{m_1} (\ln x)^{k-1}$
  in the general solution.
- **Distinct complex roots**
  For each distinct (conjugate) pair $m = \alpha \pm i\beta$ of complex roots, there will be terms of the form
  $c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$ in the general solution.
- **Repeated complex roots**
  For each (conjugate) pair $m = \alpha \pm i\beta$ of complex roots with multiplicity k, there will be $2k$ terms of the form
  $x^{\alpha} \cos(\beta \ln x),\ (\ln x) x^{\alpha} \cos(\beta \ln x),\ (\ln x)^2 x^{\alpha} \cos(\beta \ln x), \cdots, (\ln x)^{k-1} x^{\alpha} \cos(\beta \ln x)$
  $x^{\alpha} \sin(\beta \ln x),\ (\ln x) x^{\alpha} \sin(\beta \ln x),\ (\ln x)^2 x^{\alpha} \sin(\beta \ln x), \cdots, (\ln x)^{k-1} x^{\alpha} \sin(\beta \ln x)$
  in the general solution.

See Example 5 done in class

Method 2

Solving

Cauchy Euler differential equation

using

the substitution $x = e^t$

By converting the Cauchy Euler equation

into

a linear equation with constant

Main idea of solving Cauchy Euler equation using the substitution $x = e^t$

Main Idea to solve

Use the substitution $x = e^t$ or $t = \ln x$

- Converts into a linear constant coefficient equation

New independent variable 't'

Need to convert derivatives $\frac{d^k y}{dx^k}$ into $\frac{d^k y}{dt^k}$

Conversion of derivatives

- $x\frac{dy}{dx} = \frac{dy}{dt} = Dy$
- $x^2\frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y$
- $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$\vdots$

- $x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots\big(D-(n-1)\big)y$

In all the formulas $D = \frac{d}{dt}$

See class explanation

Method of solving Cauchy Euler equation using the substitution $x = e^t$

- Given

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0 \qquad (*)$$

1 Use $x = e^t$ or $t = \ln x$ to convert (*) into linear equation with constant coefficients

- Remember to convert all $\frac{d^k y}{dx^k}$ into $\frac{d^k y}{dt^k}$
- See formulas on previous page

2 Solve the linear equation with constant coefficients

- Use methods of 4.3

3 Use $t = \ln x$ to get the answer in original variable x

See Examples 6, 7 done in class

End of 4.7
Do Qs. 1-38