Section 4.6 *Variation of parameters*

The annihilator method learnt in previous section can only be applied for a few kinds of non-homogeneous differential equations. For example, it can not be used to find y_p for simple equation like $y'' + y = \tan x$.

Here we learn a method that can be applied to more general problems.

Learning Outcomes

After completing this section, you will inshaAllah be able to

a. find particular solution of ***non-homogeneous linear*** differential equations using **variation of parameters**

Method of variation of parameters

What is it for? When can we use it? Key idea?

- A method for finding particular solution of non-homogeneous linear differential equations if we know the complimentary solution y_c
- Works for equations of all order, but here we will only focus on 2nd order equations.

What is the question?

- Given:

  A 2nd order non-homogeneous linear equation

  $$y'' + p(x)y' + q(x)y = f(x) \qquad (*)$$

  and

  its complimentary solution $y_c = c_1 y_1 + c_2 y_2$

- To find y_p so that $y = y_c + y_p$ is general solution of **(*)**

Key idea of the method

- Vary the constants c_1, c_2 in y_c

  i.e replace c_1, c_2 **with functions** $u_1(x),\ u_2(x)$ **in** y_c

- and try to find y_p of the form $y_p = u_1(x)y_1 + u_2(x)y_2$

Next we look at main question:

How to find $u_1(x),\ u_2(x)$

How to find $u_1(x),\ u_2(x)$

* If we know the complimentary solution

$$y_c = c_1 y_1 + c_2 y_2$$

of a homogeneous equation $y'' + p(x)y' + q(x)y = 0$

* then a particular solution y_p of the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is given by

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

* where the functions $u_1(x),\ u_2(x)$ can be found by integrating

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Can you see why W is never zero?

- and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Wronskian W of y_1, y_2

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}$$

1st column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$$

2nd column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

See discussion of proof in class

Procedure for solving by "Variation of Parameters"

- **Question:** To find general solution of a 2nd order non-homogeneous linear differential equation
- Write equation in the form

Important: Leading coefficient must be 1

$$y'' + p(x)y' + q(x)y = f(x) \quad (*)$$

- Find complimentary solution $y_c = c_1 y_1 + c_2 y_2$ by solving

$$y'' + p(x)y' + q(x)y = 0$$

- Set

$$y_p = u_1(x)y_1 + u_2(x)y_2 \qquad (1)$$

- Find $u_1(x)$, $u_2(x)$ by integrating $u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$

where

- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ — Wronskian W of y_1, y_2

$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}$ — 1st column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$ — 2nd column of W replaced by $\begin{matrix} 0 \\ f \end{matrix}$

- Write y_p using Equation (1)
- Write general solution of (*) as $y = y_c + y_p$

Example 1 Find the general solution of

$$4y'' + 36y = \csc 3x.$$

Solution: Done in class.

Exercise The differential equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/7}$$

has complimentary solution

$$y_c = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x.$$

Find the general solution.

End of 4.6
Do Qs. 1-26