Section 4.5 *Undetermined coefficients – Annihilator approach*

In the previous Sections, you have learnt how to solve some homogeneous linear differential equations. Here you will learn a method to solve non-homogeneous linear differential equations with constant coefficients.

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. explain the meaning of annihilator of a function
2. find annihilator of a function
3. find particular solution of ***non-homogeneous linear*** differential equations with ***constant coefficients*.**
    a. hence there general solution

Annihilator Method

What is it for?

When can we use it?

- **A method for finding particular solution** of following special types of non-homogeneous linear differential equations with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \qquad a_n \neq 0 \qquad (*)$$

where $f(x)$ is a

- polynomial
- or exponential function
- or sine function
- or cosine function
- or a combination of these

For those *f(x)* that can occur as solution of homogeneous linear equations with constant coefficients

- $y'' - 2y' + 3y = x^2 \cos 3x$
- $y'' - 2y' + 3y = -2x^3 e^{4x}$

Can use annihilator method

- $y'' - 2y' + 3y = \dfrac{1}{x}$
- $y'' - 2y' + 3y = \tan x$

Can not use annihilator method

- The method uses an operator called "Annihilator"
- Before learning method, we first learn how to find annihilator of *f(x)*

operator that kills *f(x)*

Finding annihilator of *f(x)*

- Recall a linear differential operator L with constant coefficients is of the form

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

An operator L is annihilator of function y if

$$L(y) = 0$$

i.e. y is a solution of this differential equation

f(x) of the form that can occur as solution of a homogeneous linear constant coefficient equation

- Given a function *f(x)*
- We find the differential equation

$$L(y) = 0$$

which has *y=f(x)* as solution.

- This gives annihilator L of *f(x)*.

Main Step

- Constructing differential equation from solution
- Reverse of what we did in 4.3

See Examples 1 to 9 done in class

Recall the following fact used in examples

$$\left(m-(\alpha+i\beta)\right)\left(m-(\alpha-i\beta)\right) = m^2 - m(\alpha-i\beta) - m(\alpha+i\beta) + \alpha^2 + \beta^2$$
$$= m^2 - 2m\alpha + \alpha^2 + \beta^2$$

Summary: Annihilators for different *f(x)*

- $f(x) = \left(c_0 + c_1 x + \cdots + c_{k-1} x^{k-1}\right) e^{ax}$

  $$L = (D - a)^k$$

- $f(x) = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$

  $$L = \left(D - (\alpha + i\beta)\right) \cdot \left(D - (\alpha - i\beta)\right) = \left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)$$

- $f(x) = \left(c_0 + c_1 x + \cdots + c_{k-1} x^{k-1}\right) e^{\alpha x} \cos \beta x + \left(d_0 + d_1 x + \cdots + d_{k-1} x^{k-1}\right) e^{\alpha x} \sin \beta x$

  $$L = \left(D - (\alpha + i\beta)\right)^k \cdot \left(D - (\alpha - i\beta)\right)^k = \left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^k$$

Useful Results

- If $L(y_1) = 0$, $L(y_2) = 0$ then $L(c_1 y_1 + c_2 y_2) = 0$
- If $L_1(y_1) = 0$, $L_2(y_2) = 0$ and $L_1(y_2) \neq 0$, $L_2(y_1) \neq 0$ then

  $L_1 L_2 (y_1 + y_2) = 0$.

  Proof: $L_1 L_2 (y_1 + y_2) = L_1 L_2 y_1 = L_2 L_1 y_1 = 0$

See Examples 10, 11 done in class

What next?

Next we learn how to use annihilators of $f(x)$ to find particular solutions of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \qquad a_n \neq 0 \qquad (*)$$

Key Idea

- Find annihilator of *f(x)*
- Apply it to (*)
    - this will result in a homogeneous equation
- Solve the resulting homogeneous equation

Recall
General solution of (*) is given by
$y = y_c + y_p$
where y_c is general solution of associated homogeneous equation

Solution of non-homogeneous equations using annihilator method

Given a non-homogeneous linear differential equation with constant coefficients

$$L(y) = f(x) \qquad (*)$$

f(x) of the form explained on Page 4.5_2

Find y_c

- by solving $L(y) = 0$

Find annihilator L_1 of $f(x)$

- by method learnt above

Find form of y_p

- Operate L_1 in (*) to get homogeneous equation

$$L_1(L(y)) = 0 \qquad (**)$$

- Solve (**) by methods of 4.3

  - Gives solution with arbitrary constants
  - This gives the form of y_p

  Ignore contributions from (or duplication with) y_c

Determine y_p

- Put the y_p (with arbitrary coefficients) in (*)
- Determine the unknown coefficients

See Examples 12, 13 done in class

End of 4.5