Section 4.2 *Reduction of order*

- This is a method for finding general solution of homogeneous linear 2nd order ODEs in a special situation as explained below.
- Recall that we need two linearly independent solutions to write general solution of a homogeneous linear 2nd order ODE.

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. use a known one solution of a given homogeneous linear 2nd order ODE
    - to first reduce it to a 1st order linear ODE
    - and then to solve reduced ODE to get a 2nd solution of the given homogeneous linear 2nd order ODE.

Before we proceed we look at the following fact about two linearly independent solutions

Recall from last section

- If $y_1(x)$, $y_2(x)$ are linearly dependent then

  $$c_1 y_1 + c_2 y_2 = 0$$

  has a non-trivial solution i.e. at least one of c_1, c_2 is non zero.

  - Assume $c_1 \neq 0$.
  - Then we can write $y_1 = -\frac{c_2}{c_1} y_2$.
  - Hence we have the following fact.

Two solutions are linearly dependent

$\Leftrightarrow$

they are constant multiple of each other

This fact is needed to understand the idea of reduction of order.

Reduction of order for 2nd order homogeneous linear ODEs

- A method for reducing order & getting a 2nd solution of (*).
- Requires we already know one solution of (*)

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Given $y_1(x)$ as one solution of (*)
- Aim: To find 2nd linearly independent solution $y_2(x)$

- If $y_2(x)$ is another linearly independent solution then we must have $\frac{y_2(x)}{y_1(x)} \neq$ constant.
- Hence we take $\frac{y_2(x)}{y_1(x)} = u(x)$

- Take the form of $y_2(x)$ as

$$y_2(x) = u(x)y_1(x) \qquad (**)$$

- Substituting $y_2(x)$ in (*) converts (*) to a differential equation in $u(x)$ which only involves derivatives of $u(x)$. Hence it can be reduced to a 1st order linear equation in $u(x)$.
- Solve this linear equation to get $u(x)$.
- Use (**) to get $y_2(x)$.

Terms involving $u(x)$ drop out

See examples and class explanation

See Examples 1, 2 done in class

Formula for directly finding the second solution $y_2(x)$

- Given $y'' + P(x)y' + Q(x)y = 0$ (*)
- Given a solution $y_1(x)$
- Then a second solution $y_2(x)$ is given by

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

See Examples 3 done in class

Proof

- The substitution $y_2(x) = u(x)y_1(x)$ converts (*) to

$$u''y_1 + (2y_1' + P(x)y_1)u' = 0 \qquad \text{(i)}$$

- Now the substitution $w = u'$ gives first order linear equation

$$w' + \left(2\frac{y_1'}{y_1} + P(x)\right)w = 0 \qquad \text{(ii)}$$

Solving (ii) gives

$$w = C_1 \frac{e^{-\int P(x)dx}}{(y_1)^2}$$

- From $u' = w$ we get

$$u = C_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx + C_2$$

- Choosing $C_1 = 1, C_2 = 0$ we obtain the second solution

$$y_2 = uy_1 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

End of 4.2

Do Qs: 1-20