Section 4.1 *Linear differential equations: Basic theory*

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. explain what are initial value problems & boundary value problems of n^{th} order linear differential equations, and their basic properties,
2. identify the homogeneous and non-homogeneous ODEs
3. write an ODE in operator notation
4. explain about general solutions of homogeneous linear differential equations
    a. basic facts about solutions
    b. linear independence of solutions
5. check linear independence using Wronskians
6. tell about solutions of non-homogeneous linear differential equations
    a. complimentary solution or function
    b. particular solution
    c. general solution

nth order linear differential equations: IVPs

Standard form of n[th] order linear differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \quad \textbf{(*)}$$

- Homogeneous if $g(x)=0$
- Non-homogeneous if $g(x)\neq 0$

The corresponding IVP

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \quad \textbf{(1)}$$

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\cdots\cdots,\ y^{(n-1)}(x_0)=y_{n-1} \quad (2)$$

n conditions

Example 5.2.1

1) $2e^x y''' + \left(2\sin x\right)y'' - 4x^2 y' - 5y = 3$ linear, 3[rd] order

2) $xy^{(5)} + 4y^{(4)} + xy''' = x\sin x$ linear, 5[th] order

Existence of a unique solution of IVPs

- If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ are continuous on an interval I and $\boxed{a_n(x)\neq 0}$ for all x in I.
- Then IVP (1-2) has a unique solution.

See Examples 1, 2 done in class

Boundary value problems (BVPs)

Conditions specified at different points

Example:

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$\boxed{y(a)=y_0,\ y(b)=y_1}$$

Boundary conditions

Solution

- satisfies ODE
- satisfies boundary conditions

Can have one or many or no solutions

See Example 3 done in class

Differential operator notation for ODEs

- Setting

$$D \text{ as } \frac{d}{dx},\ D^2 \text{ as } \frac{d^2}{dx^2},\ldots\ldots,D^n \text{ as } \frac{d^n}{dx^n}.$$

- An nth order differential operator can be defined as

$$\boxed{L=a_n(x)D^n+a_{n-1}(x)D^{n-1}+\cdots\cdots+a_1(x)D+a_0(x)}.$$

- This can be used to write ODEs in compact form.

See Example 4 done in class

Remaining part of this section,
presents facts and properties of solutions of

homogeneous linear equations

non-homogeneous linear equations

- Here we will only do introduction and basic facts.
- Some methods of finding solutions will be done in subsequent sections.

Facts about solutions of homogeneous linear equations

- Given $a_n(x)\dfrac{d^n y}{dx^n}+a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\dfrac{dy}{dx}+a_0(x)y=0 \qquad (*)$

If you know few solutions, how many solutions can you generate?

Facts about solution

- If $y_1(x), y_2(x)\cdots, y_k(x)$ are solutions of (*) then their linear combination
$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$
is also a solution of (*)
- If $y_1(x), y_2(x), \cdots, y_n(x)$ are n linearly independent solutions of (*), then the general solution of (*) is
$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x).$$

Superposition Principle

General Solution

So the question of finding all solutions of Eq. (*) reduces to the question of finding *n* linearly independent solutions

see next page for the **meaning of linearly independent solutions**

See Example 5 done in class

Meaning of linearly independent functions

Definition:

The functions $y_1(x),\, y_2(x), \cdots, y_n(x)$ are linearly independent if the equation

$$\boxed{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0}$$

has only solution $\boxed{c_1 = 0,\, c_2 = 0, \cdots, c_n = 0}$.

Otherwise $y_1(x),\, y_2(x), \cdots, y_n(x)$ are linearly dependent.

See Examples 6, 7, 8, 9 done in class

The examples show that checking linear independence using definition is not straightforward.

Below we see a practical method of checking linear independence

Checking linear independence of solutions of ODEs (Practical method)

Definition: The Wronskian of functions $y_1(x), y_2(x), \cdots, y_n(x)$ is defined as

$$W(y_1, y_2, \cdots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The solutions $y_1(x), y_2(x), \cdots, y_n(x)$ of nth order homogeneous linear differential equation are linearly independent on interval I

$$\Leftrightarrow$$

the Wronskian $W(y_1, y_2, \ldots, y_n) \neq 0$ for some point $x_0 \in I$.

See Examples 10, 11, 12, 13 done in class

Solutions of homogeneous linear equations (Wrap Up)

- Given $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$ (*)
- We have seen that we need to find n linearly independent solutions of (*).
- Also learnt how to check linear independenece of solutions.

We will learn in remaining chapter how to find independent solutions.

- Here we wrap up the facts covered uptil now.

A set $y_1(x), y_2(x), \cdots, y_n(x)$ of linearly independent solutions of (*) is called fundamental set of solutions of (*)

Checking linear independence

- use Wronskian

If $y_1(x), y_2(x), \cdots, y_n(x)$ is a fundamental set of solutions of (*) then

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

is a general solution of (*)

Homogeneous linear equations always have fundamental set of solutions

See Example 14 done in class

Next we focus on

general solutions of nth order

non-homogeneous linear differential equations

- Only introduction: what is the structure of the general solution of non-homogeneous linear equations
- Methods of finding solutions will be done later

General solution of non-homogeneous linear differential equations

- Given $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$ (*)
- The associated homogeneous equation is

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0 \qquad (**)$$

Determining general solution of (*) requires

- Find the general solution of associated homogeneous equation (**); denoted as y_c
- Find a particular solution of non-homogeneous equation (*); denoted as y_p
- The general solution of (*) is given by $y=y_c+y_p$.

Complementary solution or Complimentary function

Particular solution

General Solution

See Example 15 done in class

It can be shown that all solutions of (*) can be obtained from $y=y_c+y_p$

Remember

To find general solution of non-homogeneous linear equation

- we need to find general solution of associated homogeneous equation
- we need to find only one particular solution of non-homogeneous equation

Superposition principle for non-homogeneous linear equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be particular solutions of

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x) \quad \text{for } i = 1, 2, \ldots, k.$$

Then

$y_p = c_1 y_{p_1} + c_2 y_{p_2} + \ldots + c_k y_{p_k}$ is a particular solution of

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_1(x)\frac{dy}{dx} + a_0(x)y =$$

$$= c_1 g_1(x) + c_2 g_2(x) + \cdots + c_k g_k(x)$$

See Example 16 done in class

End of 4.1

Do Qs. 1-36