Section 3.1 *Linear Models*

In this section, we will model some simple real-life situations which give rise to linear differential equations.

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. explain mathematical models for studying
   a. Cooling or warming problems
   b. Series circuits problems
2. solve these models

What is modeling with ODEs

It is the process of writing a differential equation to describe a physical situation involving rate of change

- Along with modeling few situations, we will also learn the basic principles of constructing mathematical models

Why do we need to study models?

- Usually to answer the questions posed in real world situation, we need to know relation between different quantities.
- Let us look at a hypothetical situation.

Let's look at a very simple situation. Suppose you have invented a very efficient new car called "Great Mathematical Car (in short GMC)" which covers distance *x (in km)* according to the following relation between distance *x (km)* time *t (hrs)*

$$x(t) = 10t^4.$$

Then you can easily answer the following questions (since you know the relation):

i. How much distance will you cover in 2 hours?

ii. How long will it take you to cover 10,000 kms or may be how long will your car take to reach moon?

What if the information from real world situation does not give you a relation between different quantities, rather it gives you information involving rates of change of a quantity (or quantities). If you model such situations, you automatically get differential equations, as we see in examples below.

Before we get to the examples of models, note that:

Construction of model is based on

- Scientific laws (physical, biological, engineering etc)
  - Describing the situation and the rates of change involved.
- Mathematical way of describing rate of change

Cooling or warming models

- First look at an example

Example: 1 The initial temperature of a laban bottle is $28^\circ C$. You put it in your refrigerator having temperature $4^\circ C$ inside. In 15 minutes you measure the temperature of laban bottle and find that it is $22^\circ C$. How long do you have to wait so that the temperature of laban bottle is $10^\circ C$ (and you can have a cool drink!!)?

Solution: Consists of construction of model and its solution.

Construction of model

$T(t)$: temperature of laban at time t

T_{frig}: temperature of refrigerator $= 4^\circ C$

- Introducing variables
- Writing given information

Scientific principle: Newton's law of cooling/warming

The rate of change of the temperature $T(t)$ (with respect to time) of a body is proportional to the difference between T and the temperature of the surrounding medium.

Mathematically: The rate of change of population $T(t)$ with respect to time t is given by the derivative $\frac{dT}{dt}$

Combining the above we have $\frac{dT}{dt} \propto (T - T_{frig})$

Constructing model

Hence, the model representing your real life (laban) problem is given as

$$\frac{dT}{dt} = k(T - T_{frig}) \text{ with } T(0) = 28^\circ C$$

or

$$\frac{dT}{dt} = k(T - 4) \text{ with } T(0) = 28^\circ C$$

where k is the proportionality constant (depending on the surrounding medium).

Cooling or warming models (contd)

- Let $T(t)$ denote temperature of object at time t

  and T_m denote temperature of surrounding at time t.

- Then

$$\frac{dT}{dt} = k\left(T - T_m\right)$$

is the model to study **"cooling or warming problems"** according to Newton's law of cooling/warming

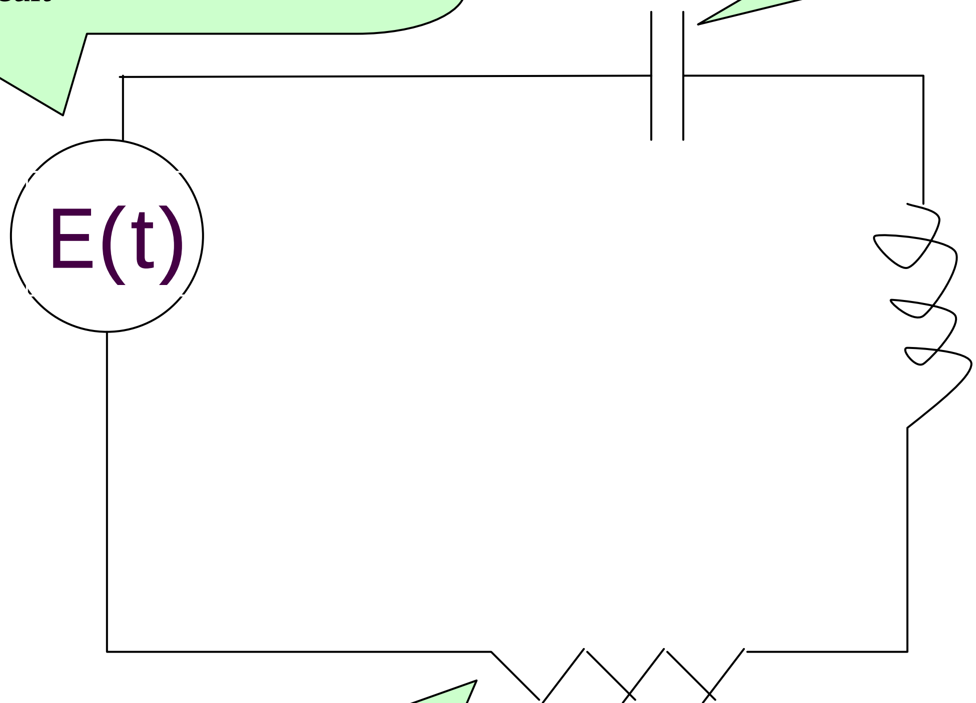
See example 2 done in class

Series circuits models

Components of a typical LRC circuit

Voltage Source
- Battery or electromotive force
- Provides voltage which produces charge to flow in the circuit

Capacitor
- Stores charge
- C for Capacitance



Resistor
- Opposes (current) flow of charge
- e.g. in heater, bulb
- R for Resistance

Inductor
- e.g. a coil of wire
- current through coil produces magnetic field
- L for Inductance

- Rate of flow of charge is called current
- Current is same at all points
- Capacitor is the only component with a charge associated to it

Series circuits models (contd.)

Laws for voltage drop across various components

- ***i(t)***: current in circuit at any time

Resistor

- Opposes current
- Voltage drop across a resistor is proportional to current

$$E_R = iR$$

Inductor

- Magnetic field of inductor opposes any change in current
- Voltage drop across an inductor is proportional to rate of change of current

$$E_L = L\frac{di}{dt}$$

Capacitor

- Voltage drop across a capacitor is proportional to the charge ***q*** on capacitor

$$E_C = \frac{1}{C}q$$

Series circuits (contd)

Scientific principle: Kirchoff's Voltage Law

The sum of voltage drops across the components of a circuit equals the applied voltage

$$E_L + E_R + E_C = E(t)$$

Model for LRC circuits

- Let $E(t)$ denote source voltage (from battery or electromotive force) at time t.
- Then

$$L\frac{di}{dt} + iR + \frac{1}{C}q = E(t) \qquad (*)$$

L: inductance

R: resistance

C: capacitance

Note:
Current *i(t)* is related to charge *q(t)* by

$$i = \frac{dq}{dt}$$

See example 3 done in class

Series circuits (contd)

Example: 4 A 200 volt electromotive force is applied to an RC series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0)=0.4$. Determine the charge as $t\to\infty$.

Solution:

- Since there is no inductor, the above equation (*) becomes

  $$iR+\frac{1}{C}q=E(t)$$

  Using $i=\dfrac{dq}{dt}$ we get

  $$R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

- For $R=1000, C=5\times10^{-6}, E=200$ we need to solve ODE

  $$1000\frac{dq}{dt}+\frac{1}{5\times10^{-6}}q=200$$

  or $$\frac{dq}{dt}+200q=\frac{1}{5}$$

  This is a simple linear equation and can be solve by methods of Section 2.3 to get the general solution

  $$q=\frac{1}{1000}+\frac{C_1}{e^{200t}} \qquad \text{(i)}$$

- Next we want to use $i(0)=0.4$ to find C_1

  By (i) we have

  $$i=\frac{dq}{dt}=-200\frac{C_1}{e^{200t}}$$

  Using $i(0)=0.4$, this gives $C_1=-\dfrac{1}{500}$.

- Solution of problem

  Using $C_1=-\dfrac{1}{500}$ in (i) we see that the solution of the problem is

  $$q(t)=\frac{1}{1000}-\frac{1}{500e^{200t}}$$

- As $t\to\infty$, we have $q(t)\to\dfrac{1}{1000}$.

End of 3.1