Section 2.5 *Solutions by subtitutions*

Certain ODE's which are neither separable nor linear can be converted to separable or linear form using appropriate substitutions. Here we are going to consider three cases of such substitutions.

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. get familiar with some techniques of transforming an ODE into simpler ODE by substitution
2. recognize and solve homogeneous 1^{st} order ODEs
3. recognize and solve Bernoulli equations

We will learn to solve three types of differential equations using substitution

Substitution Case-1: Homogeneous Equations

A homogeneous 1st order ODE is the one that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (*)$$

Examples

- $\dfrac{dy}{dx} = \dfrac{2x^2 y}{3x^3 + y^3}$
- $x\dfrac{dy}{dx} = y(\ln y - \ln x)$

Main idea to solve

Substitution $v = \dfrac{y}{x}$

- This implies $y = vx$
- Hence $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$
- Hence ODE (*) becomes separable. Why? See class discussion.

Method of solving homogeneous ODEs

- Given a 1st order homogeneous ODE. To find its solution.
- Substitute $v = \dfrac{y}{x}$ (or $y = vx$) and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (*). This transforms it into separable equation.
- Solve the separable equation.
- Use the substitution $v = \dfrac{y}{x}$ to get the general solution in the variables y and x.

See examples 1, 2, 3 done in class and the exercises discussed

Substitution Case-2: Equations of the form $\frac{dy}{dx} = f(ax+by+c)$

A differential equation of the form

$\frac{dy}{dx} = f(ax+by+c)$; $b \neq 0$ {what happens for $b = 0$?}

can always be ***reduced to a separable equation*** by the

substitution $u = ax + by + c$

See example 4 done in class

Substitution Case-3: Bernoulli Equations

A 1st order ODE of the form (for any real number n)

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n \qquad (*)$$

is called Bernoulli equation

For n=0 or n=1 this is linear, otherwise it's non-linear

Main idea to solve

Substitution $v = y^{1-n}$

transforms Bernoulli equation

into a linear equation

Ex. Why does this substitution work?

Hint: Find $\frac{dv}{dx}$ in terms of $\frac{dy}{dx}$ and plug into Eq. (*)

See examples 5, 6 done in class

End of 2.5

Do Qs. 1-30