# Section 2.4 *Exact equations*

## Learning Outcomes

After completing this section, you will inshaAllah be able to

1. know what is meant by a exact differential equations
2. check whether or not a 1st order ODE is exact
3. solve exact differential equations
4. use integrating factors to convert some non-exact equations to exact differential equations

## What are exact equations?

A 1st order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (*)$$

is exact if the expression $M(x,y)dx + N(x,y)dy$ is exact differential of some function $F(x,y)$.

i.e. $$dF(x,y) = M(x,y)dx + N(x,y)dy. \qquad (**)$$

## Q. How do we check if an equation is exact?

- Suppose that Eq. (*) is exact. The from Eq. (**)

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = Mdx + Ndy$$

- Hence
  - $$M(x,y) = \frac{\partial F}{\partial x} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} \qquad (2)$$
  - and $$N(x,y) = \frac{\partial F}{\partial y} \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} \qquad (2')$$

From Eqs. (2), (2′) we see that if an equation of the form $M(x,y)dx + N(x,y)dy = 0$ is exact then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

It can be proved that the converse is also true.

### Check for Exact Equations

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

See examples 1, 2, 3 done in class

## An important calculation procedure to study exact equations

The method of solving exact equations is based on a special computational procedure

- Given two functions $M(x,y)$ and $N(x,y)$
- How to find a function $F(x,y)$ that satisfies

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y)$$

- We learn with the help of example

See example 4 done in class

# Method of solving exact ODE's

## Main Idea

- Given an exact ODE $M(x,y)dx + N(x,y)dy = 0$.
  - i.e. $Mdx + Ndy = dF(x,y)$ for some function $F(x,y)$
  - or $Mdx + Ndy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$ for some function $F(x,y)$
- Main task is to find $F(x,y)$ so that ODE becomes $dF(x,y) = 0$
- This implies $F(x,y) = C$ is solution.

## How to find $F(x,y)$

- To find $F(x,y)$ satisfying $Mdx + Ndy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$.

To find $F(x,y)$ satisfying $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$

## Method

1. Write the ODE in the form $M(x,y)dx + N(x,y)dy = 0$ **(*)**
2. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If exact proceed as follows.
3. Get $F(x,y)$ by solving

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y)$$

4. Write the general solution by setting $F(x,y) = C$.

   Because Eq. (*) becomes $dF(x,y) = 0$.

See examples 5, 6, 7 done in class

# Converting a non-exact ODE to an exact ODE

**Main Idea**

**Multiplying by a suitable integrating factor**

- Finding integrating factor NOT always possible
- Below we discuss two cases where it is possible

- Given a non-exact ODE $M(x,y)dx + N(x,y)dy = 0$ **(*)**

## Case I

- Find $\dfrac{M_y - N_x}{N}$
- If $\dfrac{M_y - N_x}{N} = P(x)$, i.e. it is function of *x* only, then

$$u = e^{\int P(x)dx}$$

is an integrating factor for (*)

**Idea of proof similar to that done for integrating factor of Linear equations. See book for details**

Means: Multiplying (*) with u makes it an exact equation

## Case II

- Find $\dfrac{N_x - M_y}{M}$
- If $\dfrac{N_x - M_y}{M} = P(y)$, i.e. it is function of *y* only, then

$$u = e^{\int P(y)dy}$$

is an integrating factor for (*)

See examples 8, 9 done in class

*End of 2.4*

*Do Qs: 1-39*