Section 2.3 *Linear equations*

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. know what is meant by a linear 1st order differential equation
2. solve 1st order linear equations

What are linear 1^{st} order equations?

A 1^{st} order differential equation that can be written in the form

$$\frac{dy}{dx} + P(x) \cdot y = f(x) \qquad \textbf{(*)}$$

Homogeneous if $f(x) = 0$

non-homogeneous if $f(x) \neq 0$

Main idea to solve
Multiply the entire Equation (*) by an integrating factor so that the left hand side terms combine into a single derivative, and then we can integrate both sides to solve differential

Q. How do we find integrating factor for Differential Equation (*)

- Recall that $\frac{d}{dx}(uy) = uy' + u'y$
- If we multiply (*) by u, we get

$$uy' + uP(x) \cdot y = uQ(x) \qquad \textbf{(**)}$$

- Now, if we have $u' = uP(x)$ then the left hand side of (**) can be written as

$$uy' + u'y \qquad \text{or} \qquad \frac{d}{dx}(uy)$$

- So **now the remaining question for us is**

  "What should be u so that $u' = uP(x)$"

- This is easy to answer. Because $\frac{du}{dx} = uP(x)$ is separable & can be solved as

$$\frac{du}{u} = P(x)dx \quad \Rightarrow \quad \ln u = \int P(x)dx \quad \Rightarrow \quad u = e^{\int P(x)dx}$$

Hence, by multiplying the differential equation (*) with $e^{\int P(x)dx}$, we can convert L.H.S. of (*) into a single derivative.

Method of solving

Question: Given linear 1st order ODE. To find solution $y(x)$.

Step 1: Write the equation in the following (proper) form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \qquad \textbf{(*)}$$

Step 2: Find integrating factor by $u(x) = e^{\int P(x)dx}$.

Step 3: Multiply the ODE (*) throughout by integrating factor.

Step 4: Write the L.H.S. as a single derivative by using the product rule for differentiation.

Step 5: Integrate to get the general solution (in explicit or implicit form).

Step 6: In case of IVP, use initial conditions to get particular solution.

Step 7: Check your solution by taking derivatives & putting back in equation.

See Examples 1, 2, 3, 4 done in class

Note: See special trick in Examples 2.

Recall: (From Math102) Error function is an example of a function defined as integral

Standard Notation

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Needed in Example 3

How to work with such functions

Fundamental Theorem

$$\frac{d}{dx}\left(\int_a^x f(t)dt \right) = f(x)$$

End of 2.3
Do Qs. 1-37