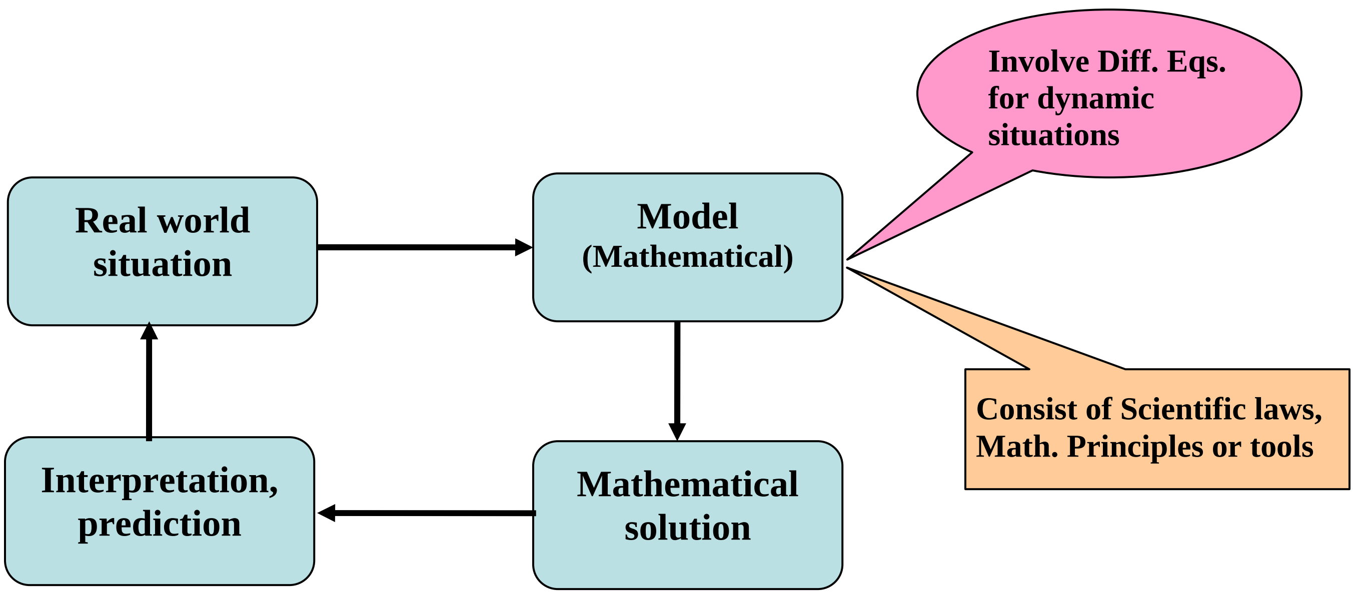
Before we start let's look at

Systematic way of investigating engineering/scientific problems



Section 1.1 *Definitions and Terminology*

Learning Outcomes

After completing this section, you will inshaAllah be able to

1. know the meaning of a differential equation
2. understand general aims of studying differential equations
3. understand the aims of this course
4. know the classification and basic terminology related to differential equations
5. understand the meaning of solution of an ODE and see how to verify the solution
6. be able to recognize different kinds of solutions
    a. explicit and implicit solutions
    b. constant solutions
    c. general and particular solutions
7. be able to plot solution curves using MATLAB

What are ordinary differential equations?

Many models of engineering systems involve rate of change. Hence, there is a need to include derivatives in mathematical models of such systems. The equations thus obtained are differential equations, whose study forms the basis of simulation of almost all continuous engineering processes. In this course we will focus only on ordinary differential equations.

Definition: An **ordinary differential equation** is an equation involving a function *y* and its derivatives.

Examples:

1. $\dfrac{dy}{dx} = \sin x$ (can you see $y = -\cos x + C$ is a solution; why?)

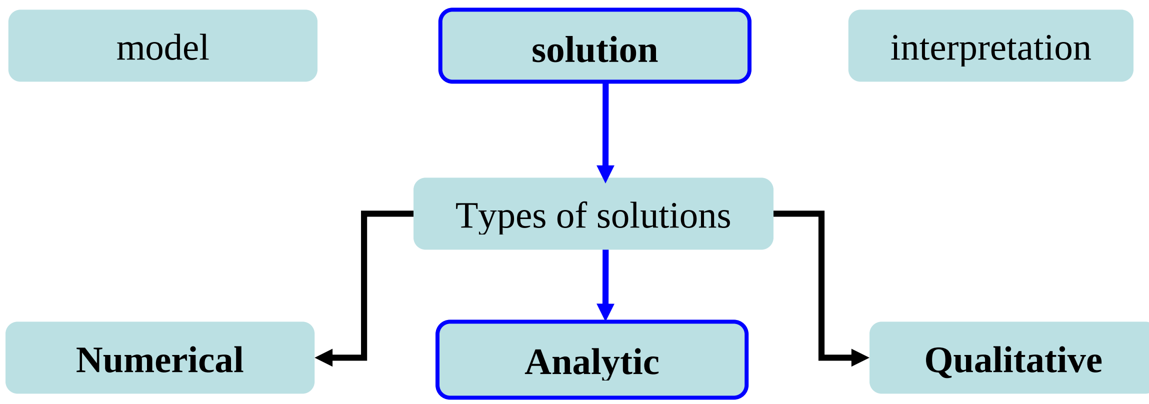
2. $\dfrac{d^2y}{dx^2} + 4y = 0$

3. $\dfrac{dy}{dt} + y^2 \sin t = 3e^{t^2}$

Can you recognize the dependent and independent variables in these equations?

General aims of studying differential equations

[For continuous dynamic situations]



Aims of this course

a) Focus on methods and techniques for finding analytic solutions for basic type of ODE's.

- o Note: In practice finding analytic solution is not always easy or possible.

b) Aim to make you familiar with some powerful and fundamental methods with that can help you to investigate many kinds of differential equations in your engineering careers.

Classification & terminology of differential equations

- **Ordinary (ODE) & partial differential equations (PDE)**

  a) $y'' + y\sin x = 1$; $y = f(x)$; ODE

  b) $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$; $u = f(x,t)$; PDE

- **Order of ODE's.** The order of ODE is the order of the highest derivative that appears in the differential equation.

  a) $y' + xy + \sin x = 2$; First order ODE

  b) $y^5 y''' + (y')^2 + e^x y = \tan x$; Third order ODE

- **General and Normal form.**

  - The most general form of an nth order ODE is
    $$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$
  - An nth order ODE is said to be in normal form if it can be written as $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$

    i.e. can be solved explicitly for the highest derivative

    a) Normal form of $4xy' + y - x = 0$ is
    $$y' = \frac{x - y}{4x}.$$

    b) The equation $(y')^2 + y^2 = 1$ can not be written in normal form.

In our course, we will be dealing with ODE's that can be written in normal form.

- **Linear and non-linear ODE's**

An ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ is said to be linear if F is a linear function of $y, y', \ldots, y^{(n)}$.

Note: it doesn't matter whether or not F is linear in x

Otherwise ODE is said to be non-linear.

a) $y'' + xy = 0$; linear

b) $y' + y^2 = 0$; non-linear

c) $2yy' + y'' = 0$; non-linear

d) $y'' + \sin y = 0$; non-linear

e) $y'' + \sin x = 0$; linear

f) $y'' + y\sin x = 0$; linear

Why?

Idea of solution of ODE and its verification

Obviously, a function that satisfies the ODE is a solution of ODE.

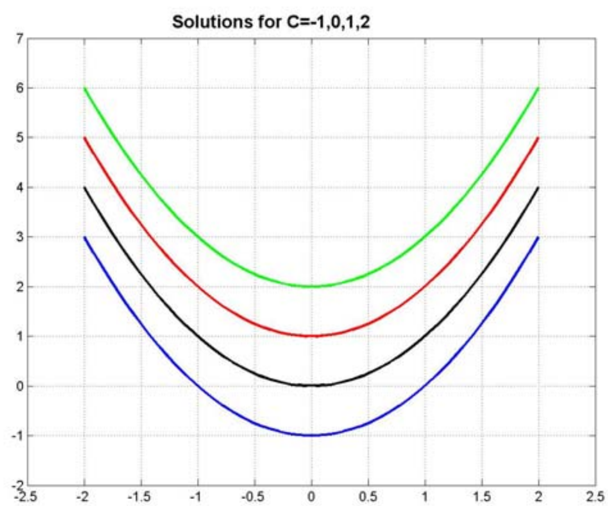
- How to verify?
- See class explanation

Example: Consider the ODE $\frac{dy}{dx} = 2x$. Then

$$y = x^2 + C \qquad (1)$$

is a solution for each value of constant *C*. {Why? Verify}

Note: For each *C* we get a different solution, so an ODE can have infinitely many solutions. The following figure shows some of them.



However, if we specify some conditions e.g. when $x = 0$ then $y = 2$ {or $y(0) = 2$ } then using Equation (1) we get

$$2 = 0 + C \quad \Rightarrow \quad C = 2,$$

hence getting a unique solution $y = x^2 + 2$.

Exercise: Verify that $y_1 = \cos 2x$, $y_2 = \sin 2x$ are both solutions of

$$y'' + 4y = 0.$$

Can you think of anymore solutions?

What kind of solutions can we get?

Explicit solutions

- of the form $y = y(x)$
- $y = x^2 + C$ is an explicit solution of $\dfrac{dy}{dx} = 2x$

Implicit solutions

- not in explicit form
- given by equation of the form $F(x, y) = 0$
- $y^2 = x^2 - C$ is an implicit solution of $\dfrac{dy}{dx} = \dfrac{x}{y}$

Constant solutions

A special type of explicit solution

- Given ODE $\dfrac{dy}{dx} = f(x, y)$.
- If $y = k$ is a solution of $f(x, y) = 0$ then obviously $y = k$ satisfies the ODE $\dfrac{dy}{dx} = f(x, y)$. [Why? See class explanation]
- Solutions $y = k$ obtained by solving $f(x, y) = 0$ are called constant solutions.
- See examples done in class.

Obviously constant solutions will not always exist. (Why?)

General solution, particular solution and solution curves

- For understanding, let's consider simple ODE's of the form

$$\frac{dy}{dx} = f(x)$$

- As you will see later, such equations can be solved by integration to get the solution

$$y = \int f(x)dx + C$$

- Because of the process of integration, the solution $y(x)$ contains arbitrary constant C. And for different values of C, we get a family of solutions of the differential equation.

- A solution of ODE containing arbitrary constant is called **general solution**.

- For particular values of C, we get a **particular solution**. Such solutions can be obtained by putting initial or other conditions. [See section 1.2 for more]

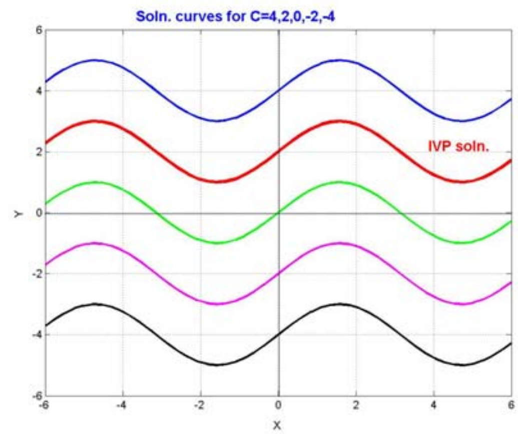
- The graphs of solutions are called **solution curves**. [See next page]

See Examples 1, 2, 3 done in class

Examples of solution curves

Example 1:

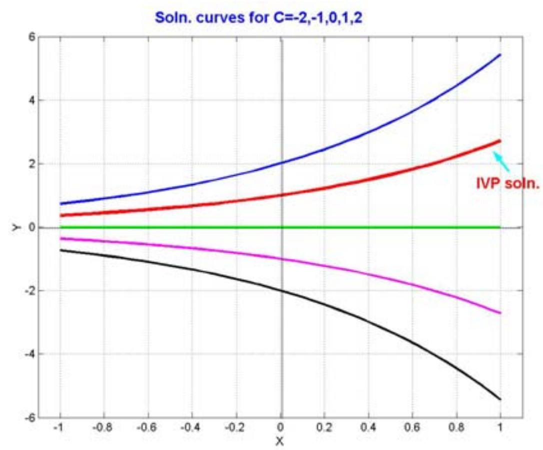
$$\frac{dy}{dx} = \cos x$$



General Solution
$y = \sin x + C$

Example 2:

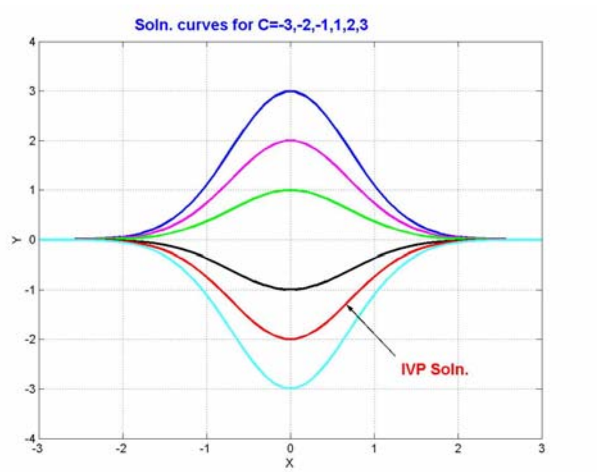
$$\frac{dy}{dx} = y$$



General Solution
$y = Ce^{x}$

Example 3:

$$\frac{dy}{dx} = -2xy$$



General Solution
$y = Ce^{-x^2}$

Plotting a curve using MATLAB

Given a function $y(x)$, how to make its graph

To plot $y = e^{-x}$ from $x = -1$ to $x = 4$, we can use the following sequence of commands.

>> x=linspace(-1,4,50);

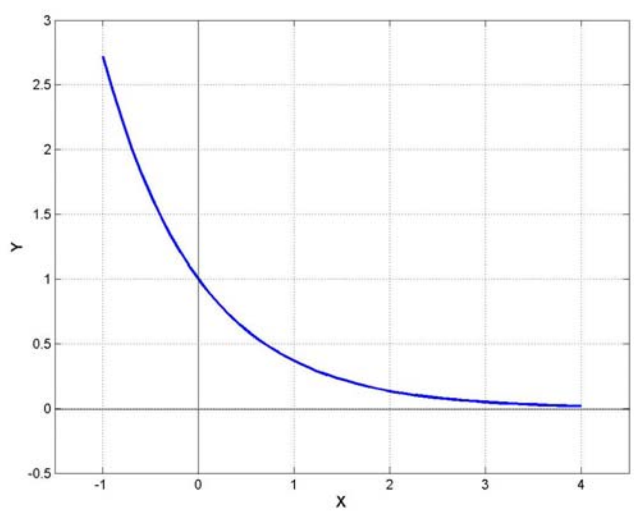
Generates a vector x with 50 equally spaces values from x=-1 to x=4

>> y=exp(-x);

Calculates the function y (in vector form) over each x-value.

>> plot(x,y)

Plots the corresponding X, Y values from the vectors x & y, and generates



Exercise: Plot $y = x^2 \cos x$ from $x = -1$ to $x = 4$.

This is a simple exercise, but if you have not worked with MATLAB earlier, you may have a slight problem. Play around and you should be able to resolve it otherwise you know all the means of reaching me.

Check how to use other related commands like xlabel, ylabel by using
>> help

Plotting multiple curves using MATLAB

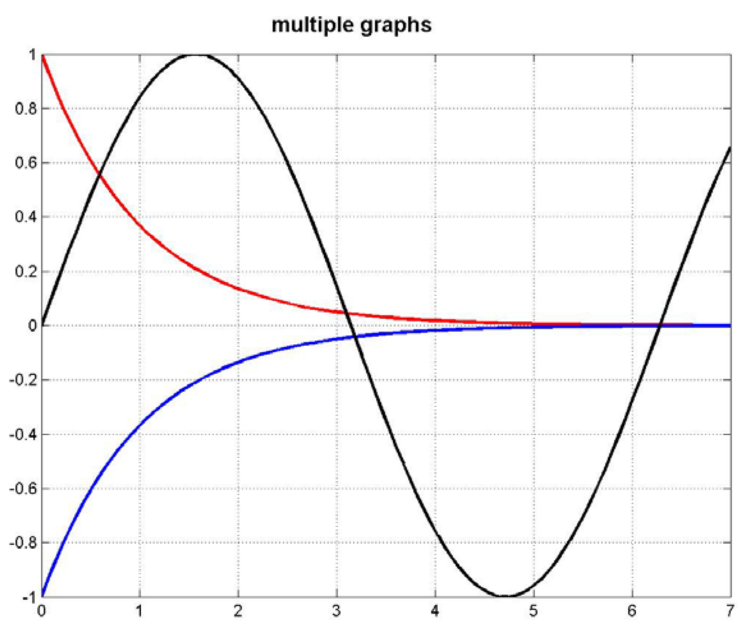
Each time you make a new graph, MATLAB erases the previous one and draws the new one. If you want to have more plots on the same graph-window, one way is to use the commands "hold on" and "hold off" as explained below.

Example: Plot $y_1 = e^{-x}$, $y_2 = -e^{-x}$, $y_3 = \sin x$ from $x = 0$ to $x = 7$ (on same graph-window).

Solution: Using the following sequence of commands (see explanation in class)

```
>> x=linspace(0,7,100);
>> y1=exp(-x);
>> plot(x,y1)
>> hold on
>> y2=-exp(-x);
>> plot(x,y2)
>> y3=sin(x);
>> plot(x,y3)
>>hold off
```

we get the following figure.



End of 1.1
Do Qs. 1-34