1. (12 points) Solve the initial-value problem

$$x^2 \frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}.$$

2. (12 points) Solve $y^{(4)} - 9y'' = e^{3x}$.

3. (15 points) Consider the differential equation

$$xy'' - y' - 2y = 0.$$

(a) Show that $x = 0$ is a regular singular point for the differential equation.

(b) Find a series solution about $x = 0$.

4. (12 points) Solve the system $X' = AX$ where $A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 4 \\ 0 & -4 & 2 \end{pmatrix}$. (Write your answer in terms of real functions)

5. (10 points) Consider the system of differential equations

$$X' = AX + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

(a) Given that $X_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ and $X_2(t) = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}$ are solutions of the associated homogeneous system. Show that they form a fundamental set of solutions of the associated homogeneous system.

(b) Form a fundamental matrix of the system.

(c) Find a particular solution of the given non-homogeneous system.

6. (9 points) Let $A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -2 & 2 \\ -2 & -1 & 1 \end{pmatrix}$.

(a) Show that $A^3 = 0$. (So that $A^k = 0$ for $k \geq 4$)

(b) Compute e^{At}.

(c) Use e^{At} to find the solution of the initial-value problem

$$X' = AX, \quad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$