Section 2.3 *Calculating limits using the limit laws*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. use basic laws of limits to compute limits
2. compute limits using some practical methods
    a. direct substitution
    b. factorization and cancellation
    c. rationalization
    d. simplification
3. use above methods to compute one-sided limits and limits of piece-wise functions
4. use Squeeze theorem to find limits of special type of functions

Basic limit laws for computing limits

1) $\lim_{x \to a} c = c$.

2) $\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

3) $\lim_{x \to a} \left[f(x) \cdot g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

4) $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$

Obviously these laws are valid when the limits of all the functions involved exist.

5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

6) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n$ n: positive integer

7) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ n: positive integer

See example 1 done in class

We will keep these laws in mind but, to compute limits, we will mainly use the practical ways explained below

Practical techniques of computing finite limits

Direct Substitution

See example 2 done in class

- What happens if we try direct substitution for $\lim_{x\to 1}\frac{x^3-1}{x-1}$.
- We get $\left(\frac{0}{0}\right)$ form
- **In such situations try one of the following**

Recall from Section 2.2

If direct substitution gives

$\left(\frac{k}{0}\right)$ form (with $k \neq 0$)

then we get infinite limits.

Factorization & Cancellation

See examples 3, 4 done in class

Rationalization

See example 5 done in class

Hint
Radical sign & (0/0) form

Simplification

See examples 6, 7 done in class

Combination of above techniques

See example 8 done in class

Computing one-sided limits and limits of piece-wise functions

- Above techniques of limits are also valid for calculating one-sided limits
- Hence, can also be used to find limits of piece-wise functions

See Examples 9, 10, 11 done in class

Recall the following needed in examples

- The greatest integer function is defined as

  $[\![x]\!] =$ largest integer $\leq x$

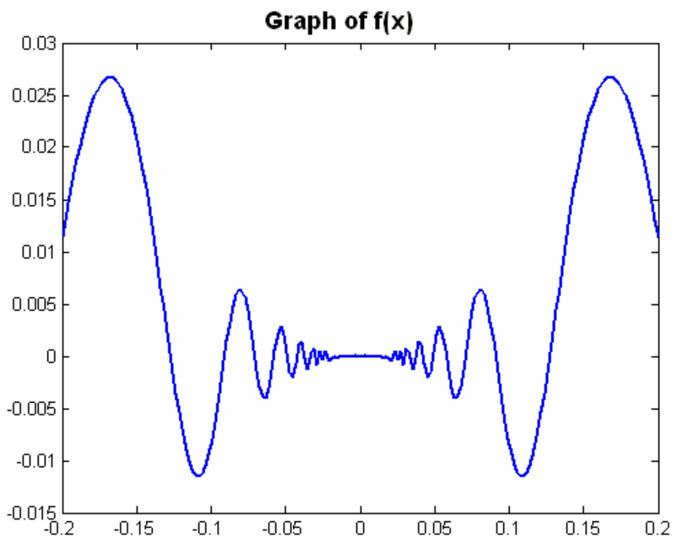
- For example
  - $[\![2.4]\!] = 2$
  - $[\![2]\!] = 2$
  - $[\![1.9]\!] = 1$

The Squeeze Theorem (a tool for finding limits in special situations)

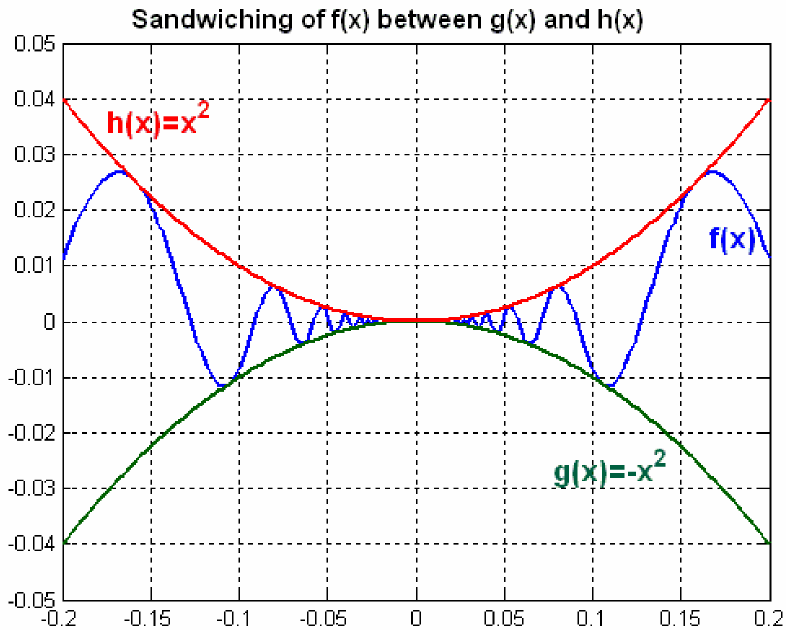
Graphical explanation

Look at $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

- Graph of $f(x) = x^2 \cos\left(\frac{1}{x}\right)$



- Squeezing of $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ between $g(x) = -x^2$ and $h(x) = x^2$.



- As $x \to 0$ we see that $h(x) \to 0$ and $g(x) \to 0$.
- Since (from graph) $f(x)$ is squeezed between $h(x)$ and $g(x)$ we must have $f(x) \to 0$ as $x \to 0$.

The Squeeze Theorem (a tool for finding limits in special situations)

How to apply it to solve questions?

Squeeze Theorem

If

$$g(x) \le f(x) \le h(x) \qquad (1)$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \qquad (2)$$

then

$$\lim_{x\to a} f(x) = L$$

Also called Sandwich Theorem

Main step needed for calculations

- Finding the appropriate Sandwiching Functions satisfying (1) & (2).
- We learn it by doing examples.

See examples 12, 13 done in class

End of 2.3.

This an important section so try to absorb the material by solving more problems.