Q.1 a) i) Find the Cartesian equation of the curve whose parametric equations are given by $x = 2\cot t$, $y = 2\sin^2 t$.

ii) Find the point where the curve intersects the y-axis.

Solution:

i) $y = 2\sin^2 t$ or $\sin^2 t = y/2$.

$x = 2\cot t$ implies that $x^2 = 4\cot^2 t = \dfrac{4\cos^2 t}{\sin^2 t}$

$$\text{or} \qquad x^2 = \frac{4(1 - y/2)}{y/2} = \frac{4(2-y)}{y}$$

Thus, the cartesian equation of the curve is $x^2 y = 8 - 4y$

ii) The curve intersects the y-axis at $(0, 2)$.

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b) Find the points on the curve $x = 6t - t^3$, $y = 3t^2$ where the tangent is parallel to the line with equation $y = 5 - 2x$.

Solution:

$$\frac{dx}{dt} = 6t - t^3; \qquad \frac{dy}{dt} = 3t^2$$

Slope of the given curve is

$$\frac{dy}{dx} = \frac{6t}{6 - 3t^2}$$

Slope of the line $y = 5 - 2x$ is equal to -2. Both slopes are equal. So, we have

$$\frac{6t}{6-3t^2} = -2 \qquad \text{or} \qquad 6t^2 - 6t - 12 = 0 \qquad \text{or} \qquad t^2 - t - 2 = 0$$

$$t = -1, 2$$

At $t = -1$: point $(-5, 3)$

At $t = 2$: point $(4, 12)$.

Q.2 Consider the polar equation $r = 2 + 2\cos 2\theta$.

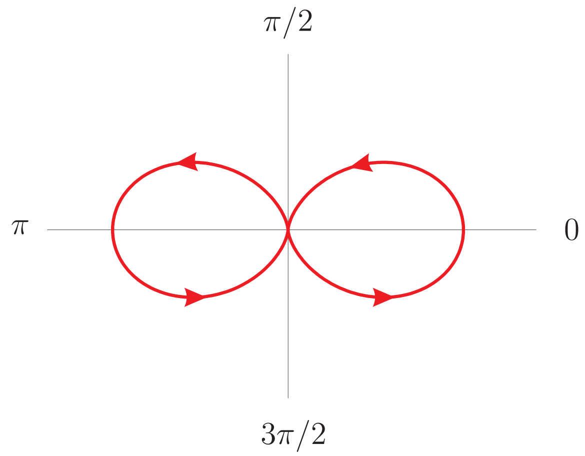
i) Sketch the curve of the given polar equation

ii) Find the slope of the tangent line to the polar curve at $\theta = \pi/4$

Solution:

i)

θ	$0 \to \frac{\pi}{4}$	$\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$\frac{3\pi}{4} \to \pi$	$\pi \to \frac{5\pi}{4}$	$\frac{5\pi}{4} \to \frac{3\pi}{2}$	$\frac{3\pi}{2} \to \frac{7\pi}{4}$	$\frac{7\pi}{4} \to 2\pi$
r	$4 \to 2$	$2 \to 0$	$0 \to 2$	$2 \to 4$	$4 \to 2$	$2 \to 0$	$0 \to 2$	$2 \to 4$



ii) $$\frac{dr}{d\theta} = -4\sin 2\theta, \quad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = -4\sin 2\theta\cos\theta - (2+2\cos 2\theta)\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = -4\sin 2\theta\sin\theta - (2+2\cos 2\theta)\cos\theta$$

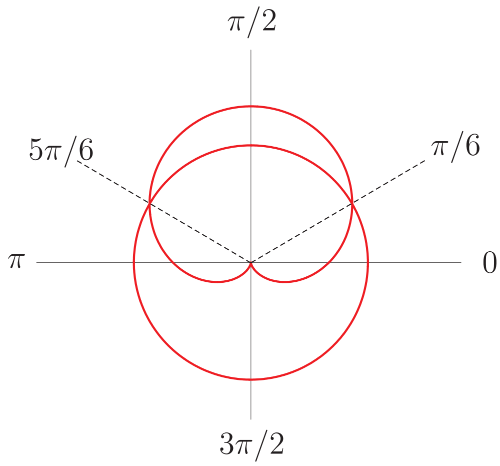
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-4\sin 2\theta\sin\theta - (2+2\cos 2\theta)\cos\theta}{-4\sin 2\theta\cos\theta - (2+2\cos 2\theta)\sin\theta}$$

At $\theta = \frac{\pi}{4}$, $$\frac{dy}{dx} = \frac{-4\sin\pi/2\sin\pi/4 - (2+2\cos\pi/2)\cos\pi/4}{-4\sin\pi/2\cos\pi/4 - (2+2\cos\pi/2)\sin\pi/4}$$

$$= \frac{-4(1)(\sqrt{2}/2) + (2)((\sqrt{2}/2))}{-4(1)(\sqrt{2}/2) - (2)((\sqrt{2}/2))} = \frac{-4+2}{-4-2} = \frac{1}{3}$$

Q.3 a) Find the area of the region inside the polar curve $r = 2 + 2\sin\theta$ and outside the curve $r = 3$.

Solution:



Points of intersection of the curves:

$$3 = r = 2 + 2\sin\theta \implies \sin\theta = 1/2 \implies \theta = \pi/6, 5\pi/6.$$

By symmetry; $2(\pi/6 \le \theta \le \pi/2)$

$$\begin{aligned}
A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}[(2+2\sin\theta)^2 - 9]d\theta &= \int_{\pi/6}^{\pi/2} [4\sin^2\theta + 8\sin\theta - 5]d\theta \\
&= \int_{\pi/6}^{\pi/2} \left[4\left(\frac{1-\cos 2\theta}{2}\right) + 8\sin\theta - 5\right] d\theta \\
&= \int_{\pi/6}^{\pi/2} [-2\cos 2\theta + 8\sin\theta - 3]d\theta \\
&= \big[-\sin 2\theta - 8\cos\theta - 3\theta\big]_{\pi/6}^{\pi/2} = -\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2}
\end{aligned}$$

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b) Set up an integral to find the area of one loop of the rose $r = 3\cos 6\theta$. (**Do Not Evaluate the Integral**)

Solution:

$$0 = r = 3\cos 6\theta \implies 6\theta = \pm\frac{\pi}{2} \implies \theta = \pm\frac{\pi}{12}.$$

$$\text{Area of one loop} = \frac{1}{2}\int_{-\pi/12}^{\pi/12} 9\cos^2 6\theta d\theta$$

Q.4 A sphere has equation $x^2+y^2+z^2 = 10y-16z+C$, where C is a constant.

i) Find the center of the sphere

ii) Find the radius of the sphere in terms of C.

iii) If the radius of the sphere is equal to 10, find the points where the sphere intersects the y-axis.

Solution:

i) $x^2 + (y-5)^2 + (z+8)^2 = 25 + 64 + C = 89 + C$.

The center of the sphere is $(0, 5, -8)$.

ii) Radius of the sphere is $\sqrt{89+C}$

iii) $\sqrt{89+C} = 10 \implies C = 100 - 89 = 11$.

Equation becomes $x^2 + y^2 + z^2 = 10y - 16z + 11$.

The sphere intersects the y-axis when $x = z = 0$

So that $y^2 - 10y - 11 = 0$.

This gives $(y-11)(y+1) = 0$ i.e. $y = 11,\ y = -1$.

The required points are $(0, 11, 0)$ and $(0, -1, 0)$.

Q.5 Let $\vec{a} = \langle \sqrt{2}, 1, 1 \rangle$ and $\vec{b} = \langle -\sqrt{2}, 4, -1 \rangle$ be two vectors in $\mathbb{R}^3$.

i) Find the scalar projection and vector projection of $\vec{b}$ onto $\vec{a}$.

ii) Find the angle between the vectors $\vec{a}$ and $\vec{a} + \vec{b}$.

iii) If $\vec{r} = \langle x, y, z \rangle$, show that the vector equation $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ represents a sphere.

Solution:

i)

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-2+4-1}{\sqrt{2+1+1}} = \frac{1}{2}$$

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{1}{4}\langle \sqrt{2}, 1, 1 \rangle = \langle \frac{\sqrt{2}}{4}, \frac{1}{4}, \frac{1}{4} \rangle$$

ii) $\vec{a} + \vec{b} = \langle 0, 5, 0 \rangle$

$$\vec{a} \cdot (\vec{a} + \vec{b}) = |\vec{a}||\vec{a} + \vec{b}| \cos\theta$$

$$5 = (2)(5)\cos\theta$$

$$\cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

iii)

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

$$(x - \sqrt{2})(x - \sqrt{2}) + (y-1)(y-4) + (z-1)(z+1) = 0$$

$$x^2 - 2 + y^2 + 5y + 4 + z^2 - 1 = 0$$

$$x^2 + (y - 5/2)^2 + z^2 = \frac{21}{4}$$

This is a sphere with radius $\sqrt{21}/2$ and center $(0, 5/2, 0)$.

Q.6 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

<u>Solution:</u>

Let

$$\vec{a} = \vec{PQ} = \langle -3, 1, -7 \rangle$$

$$\vec{b} = \vec{PR} = \langle 0, -5, -5 \rangle$$

Area of triangle $PQR = \frac{1}{2}|\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= -40\vec{i} - 15\vec{j} + 15\vec{k}$$

$$= 5\langle -8, -3, 3 \rangle$$

$$|\vec{a} \times \vec{b}| = 5\sqrt{64 + 9 + 9} = 5\sqrt{82}$$

Area of the triangle is $\dfrac{5}{2}\sqrt{82}$