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LONG TIME BEHAVIOR FOR A VISCOELASTIC PROBLEM WITH A POSITIVE DEFINITE KERNEL

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ABSTRACT. We study the asymptotic behavior of solutions for an integro-differential problem which arises in the theory of viscoelasticity. It is proved that solutions go to rest in an exponential manner under new assumptions on the relaxation function in the memory term. In particular, we consider a new family of kernels which are not necessarily decreasing.

Key words and phrases: Exponential decay, Memory term, Relaxation function, Viscoelasticity.

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1. INTRODUCTION

We shall consider the following wave equation with a temporal non-local term

$$\begin{cases} u_{tt} - \Delta u_{tt} = \Delta u - \int_0^t h(t-s)\Delta u(s)ds, \ in\ \Omega \times \mathbf{R}_+ \\ u = 0, \ on\ \Gamma \times \mathbf{R}_+ \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ in\ \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given initial data and the (nonnegative) relaxation function $h(t)$ will be specified later on. The equation in 1.1 describes the equation of motion of a viscoelastic body with fading memory (see [6],[21]). The memory term, represented by the convolution term in the equation, expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t.

Global existence and uniform decay of solutions have been discussed for similar and related problems in [2] - [5], [7] - [14], [16] - [20], [23] - [24] and in a general setting in [15]. In all these works the kernels were assumed to be of an exponential form $e^{-\beta t}$, $\beta > 0$, singular (in a neighborhood of zero) $t^{-\alpha}$, $0 < \alpha < 1$ (in this case no rate of convergence was found), of the form $t^{-\alpha}e^{-\beta t}$, $\beta > 0$, $0 < \alpha < 1$, summable functions satisfying $h'(t) \leq -\xi h(t)$, for all $t \geq 0$ or $-\xi_1 h(t) \leq h'(t) \leq -\xi_2 h(t)$, for all $t \geq 0$ for some positive constants ξ, ξ_1 and ξ_2.

The memory term produces a (weak) dissipation which drives the system to rest. In some works an exponential decay rate was proved and in some other works only convergence results were established.

In [12], the present author (with M. Medjden) proved an exponential decay result for a similar problem. The commonly used assumptions mentioned above were somewhat relaxed to non-increasing kernels satisfying $e^{\alpha t}h(t) \in L^1(0,\infty)$ for some $\alpha > 0$. The authors introduced a new functional and used the modified energy method. In turn, this result was improved in [13]. Exponential decay has been proved under the only assumption that $[h'(t) + \xi h(t)]\, e^{\alpha t} \in L^1(0,\infty)$ for some $\alpha, \xi > 0$. In particular, no decreasingness of the kernel was imposed.

In this paper, we prove exponential decay of solutions for problem 1.1 under new assumptions on the kernel $h(t)$. We do not assume any condition on the derivative of $h(t)$ and consider functions $h(t)$ such that $e^{\alpha t}h(t) \in L^1(0,\infty)$ and $e^{\frac{\alpha}{2}t}h(t)$ is (strongly) positive definite for some $\alpha > 0$. The argument is different from the previous ones. It makes use of a crucial lemma (see Lemma 2 below) and a Lyapunov type functional which was introduced for the first time by the author in [23].

The well posedness is standard. Using the Faedo Galerkin method one can prove the global existence of a weak solution to problem 1.1 (see for instance [3], [4], [14], [15]).

Theorem 1.1. *Let $u_0, u_1 \in H_0^1(\Omega)$ and $h(t)$ be a nonnegative summable kernel. Then there exists at least one weak solution u to problem 1.1 such that*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)), \ u_t \in L^\infty\left(0,\infty; H_0^1(\Omega)\right), \ u_{tt} \in L^2\left(0,\infty; H_0^1(\Omega)\right).$$

In Section 2 we prepare some material needed to prove our result. Section 3 is devoted to the statement and proof of the exponential decay result for (strongly) positive definite kernels.

2. PRELIMINARIES

We define the (classical) energy by

$$E(t) = \int\limits_{\Omega} \left(\frac{1}{2} |u_t|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla u_t|^2 \right) dx.$$

Then by the equation 1.1 $_1$ it is easy to see that

$$E'(t) = \int\limits_{\Omega} \nabla u_t \int\limits_{0}^{t} h(t-s) \nabla u(s) ds dx. \tag{2.1}$$

Observe that $E'(t)$ is of an undefined sign. Consider the modified energy

$$V(t) = E(t) - \varepsilon \Phi(t) + \Psi(t) \tag{2.2}$$

with

$$\Phi(t) = \int\limits_{\Omega} u_t u dx + \int\limits_{\Omega} \nabla u_t \nabla u dx, \tag{2.3}$$

and

$$\Psi(t) := \int\limits_{\Omega} \int\limits_{0}^{t} H_\alpha(t-s) \left(\eta |\nabla u|^2 + \mu |\nabla u_t|^2 \right) ds dx \tag{2.4}$$

where $H_\alpha(t) := e^{-\alpha t} \int\limits_{t}^{+\infty} h(s) e^{\alpha s} ds$ for some $\varepsilon, \eta, \mu > 0$ to be determined.

Proposition 2.1. *There exists $\xi > 0$ and $\varepsilon_0 > 0$ such that $V(t) \geq \xi E(t)$ for all $t \geq 0$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. : By the inequalities

$$\int\limits_{\Omega} u_t u dx \leq \frac{1}{2} \int\limits_{\Omega} |u_t|^2 dx + \frac{C_p}{2} \int\limits_{\Omega} |\nabla u|^2 dx$$

and

$$\int\limits_{\Omega} \nabla u_t \nabla u \leq \frac{1}{2} \int\limits_{\Omega} |\nabla u_t|^2 dx + \frac{1}{2} \int\limits_{\Omega} |\nabla u|^2 dx$$

where C_p is the Poincaré constant, we have

$$V(t) \geq \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right) \int_\Omega |u_t|^2 dx + \left(\tfrac{1}{2} - \tfrac{\varepsilon}{2}\right) \int\limits_{\Omega} |\nabla u_t|^2$$
$$+ \left(\tfrac{1}{2} - \tfrac{\varepsilon C_p}{2} - \tfrac{\varepsilon}{2}\right) \int_\Omega |\nabla u|^2 dx + \Psi(t).$$

Therefore, if $\varepsilon < \frac{1}{1+C_p}$ we obtain $V(t) \geq \xi E(t)$ for some constant $\xi > 0$. ∎

The following inequality will be used repeatedly in the sequel.

Lemma 2.2. *We have*

$$ab \leq \delta a^2 + \frac{b^2}{4\delta}, \; a, b \in \mathbf{R}, \; \delta > 0.$$

Definition 2.1. We say that a function $k \in L^1_{loc}[0,+\infty)$ is positive definite if

$$\int_0^t w(s) \int_0^s k(s-z)w(z)dzds \geq 0, \ t \geq 0$$

for every $w \in C[0,+\infty)$.

Definition 2.2. A function $k(t)$ is said to be strongly positive definite if there exists a positive constant σ such that the mapping $t \to k(t) - \sigma e^{-t}$ is positive definite.

For $w \in C([0,T]; L^2(\Omega))$, $T > 0$, we define

$$Q(w,t,k) := \int_0^t \int_\Omega w(s,x) \int_0^s k(s-\tau)w(\tau,x)d\tau dxds, \ \forall t \ \in [0,T].$$

By Δ_h and D_h we denote the expressions, for $T > 0$ and $0 < h < T$

$$\Delta_h w(x,t) := w(x,t+h) - w(x,t), \ x \in \Omega, t \in [0,T-h],$$

and

$$(D_h w)(x,t) := \int_0^t \Delta_h w(x,s)ds, \ x \in \Omega, \ t \in [0,T-h]$$

for every $w \in C([0,T]; L^2(\Omega))$ respectively.

The next lemma is a consequence of Lemma 2.4 in [1], the proof of which uses Lemma 2.5 of [11]. This latter lemma in turn is based on an inequality in [22] Lemma 4.2. We state it here together with a direct proof.

Lemma 2.3. *For any strongly positive definite function $k(t)$ there exists a constant $M > 0$ such that*

$$\begin{gathered}\int_\Omega w^2(x,t)dx + \int_0^t \int_\Omega w^2(x,s)dxds \\ \leq M\int_\Omega w^2(x,0)dx + MQ(w,t,k(t)) + M\liminf_{h\to 0}\frac{1}{h^2}Q(\Delta_h w,t,k(t)),\end{gathered}$$

$\forall t \in [0,T]$, *for every* $w \in C([0,T]; L^2(\Omega))$.

Proof. In the definition of $Q(\Delta_h w, t, e^{-t})$ we perform an integration by parts, we find

$$\begin{aligned}Q(\Delta_h w,t,e^{-t}) = &\int_\Omega (D_h w)(x,t)\int_0^t e^{-(t-\tau)}\Delta_h w(x,\tau)d\tau dx \\ &+\int_0^t\int_\Omega (D_h w)(x,s)\int_0^s e^{-(s-\tau)}(\Delta_h w)(x,\tau)d\tau dxds, \\ &-\frac{1}{2}\int_\Omega (D_h w)^2(x,t)dx.\end{aligned} \tag{2.5}$$

A second integration by parts in the second term on the right hand side of 2.5 yields

$$\begin{aligned}Q(\Delta_h w,t,e^{-t}) = &\frac{1}{2}\int_\Omega (D_h w)^2(x,t)dx - \int_\Omega (D_h w)(x,t)\int_0^t e^{-(t-\tau)}D_h w(x,\tau)d\tau dx \\ &-\int_0^t\int_\Omega (D_h w)(x,s)\int_0^s e^{-(s-\tau)}(D_h w)(x,\tau)d\tau dxds + \int_0^t\int_\Omega (D_h w)^2(x,s)dxds.\end{aligned} \tag{2.6}$$

Next, dividing both sides of 2.6 by h^2 and passing to the limit as $h \to 0$, it appears that

$$\begin{aligned}
&\liminf_{h\to 0} \tfrac{1}{h^2} Q(\Delta_h w, t, e^{-t}) \\
&= \tfrac{1}{2}\int_\Omega [w(x,t)-w(x,0)]^2\,dx + \int_0^t\int_\Omega [w(x,s)-w(x,0)]^2\,dxds \\
&\quad -\int_\Omega [w(x,t)-w(x,0)]\int_0^t e^{-(t-\tau)}[w(x,\tau)-w(x,0)]\,d\tau dx \\
&\quad -\int_0^t\int_\Omega [w(x,s)-w(x,0)]\int_0^s e^{-(s-\tau)}[w(x,\tau)-w(x,0)]\,d\tau dxds
\end{aligned} \tag{2.7}$$

After developing the right-hand side of 2.7 we obtain

$$\begin{aligned}
&\liminf_{h\to 0} \tfrac{1}{h^2} Q(\Delta_h w, t, e^{-t}) \\
&= \tfrac{1}{2}\int_\Omega w^2(x,t)dx + \tfrac{1}{2}\int_\Omega w^2(x,0)dx + \int_0^t\int_\Omega w^2(x,s)dxds \\
&\quad -\int_\Omega w(x,t)\int_0^t e^{-(t-s)}w(x,s)dsdx - \int_\Omega w(x,t)w(x,0)e^{-t}dx \\
&\quad -Q(w,t,e^{-t}) - \int_0^t\int_\Omega w(x,s)w(x,0)e^{-s}dxds.
\end{aligned} \tag{2.8}$$

Clearly by the algebraic inequality in Lemma 2.2, we have

$$\left|\int_\Omega w(x,t)\int_0^t e^{-(t-s)}w(x,s)dsdx\right| \le \lambda\int_\Omega w^2(x,t)dx + \frac{1}{8\lambda}\int_0^t\int_\Omega w^2(x,s)dxds, \tag{2.9}$$

$$\left|\int_\Omega w(x,t)w(x,0)e^{-t}dx\right| \le \gamma\int_\Omega w^2(x,t)dx + \frac{1}{4\gamma}\int_\Omega w^2(x,0)dx \tag{2.10}$$

and

$$\left|\int_0^t\int_\Omega w(x,s)w(x,0)e^{-s}dxds\right| \le \eta\int_0^t\int_\Omega w^2(x,s)dxds + \frac{1}{8\eta}\int_\Omega w^2(x,0)dx. \tag{2.11}$$

Gathering these estimates 2.9 - 2.11 we entail from 2.8 that

$$\begin{aligned}
&\left(\tfrac{1}{2}-\lambda-\gamma\right)\int_\Omega w^2(x,t)dx + \left(1-\eta-\tfrac{1}{8\lambda}\right)\int_0^t\int_\Omega w^2(x,s)dxds \\
&\le \left(-\tfrac{1}{2}+\tfrac{1}{4\gamma}+\tfrac{1}{8\eta}\right)\int_\Omega w^2(x,0)dx + Q(w,t,e^{-t}) \\
&\quad + \liminf_{h\to 0}\tfrac{1}{h^2}Q(\Delta_h w, t, e^{-t}).
\end{aligned}$$

It is easy to see that we may choose positive constants λ, γ and η such that the factors $\left(\frac{1}{2}-\lambda-\gamma\right)$, $\left(1-\eta-\frac{1}{8\lambda}\right)$ and $\left(-\frac{1}{2}+\frac{1}{4\gamma}+\frac{1}{8\eta}\right)$ are positive (pick for instance, $\lambda = \frac{2}{5}$, $\gamma = \frac{1}{11}$ and $\eta = \frac{1}{16}$).

The conclusion follows from the strong positive definiteness of the kernel since then there exists a positive constant σ such that

$$Q(w,t,e^{-t}) \le \sigma^{-1}Q(w,t,k(t)) \;\; for\ all\ t \in [0,T].$$

∎

3. ASYMPTOTIC BEHAVIOR

In this section we state and prove our result. First, we suppose that the kernel $h(t)$ is a $C^1(\mathbf{R}_+, \mathbf{R}_+)$ function satisfying

$(H1)$ $e^{\alpha t}h(t) \in L^1(\mathbf{R}_+)$ for some $\alpha > 0$.

$(H2)$ $e^{\frac{\alpha}{2}t}h(t)$ is strongly positive definite with $\sigma \geq 4$.

In $(H2)$, σ is as in Definition 2.2. We denote by $\bar{h}_\alpha$ the value

$$\bar{h}_\alpha := \int_0^\infty e^{\alpha t}h(t)dt.$$

Theorem 3.1. *Assume that the hypotheses $(H1)-(H2)$ hold. Then the energy of 1.1 decays to zero exponentially, that is, there exist positive constants C and $\beta > 0$ such that*

$$E(t) \leq Ce^{-\beta t},\ t \geq 0$$

provided that $\bar{h}_\alpha$ is sufficiently small.

Proof. : A differentiation of $V(t)$ (see 2.2 and 2.3) with respect to t along solutions of 1.1 gives

$$\begin{aligned} V'(t) = &\int_\Omega \nabla u_t \int_0^t h(t-s)\nabla u(s)dsdx - \varepsilon\int_\Omega |u_t|^2\,dx - \varepsilon\int_\Omega |\nabla u_t|^2\,dx \\ &-\varepsilon\int_\Omega \nabla u\int_0^t h(t-s)\nabla u(s)dsdx + \varepsilon\int_\Omega |\nabla u|^2\,dx + \Psi'(t). \end{aligned} \tag{3.1}$$

Multiplying $V(t)$ by $e^{\alpha t}$ and differentiating with respect to $t,$ we get

$$\frac{d}{dt}\left[e^{\alpha t}V(t)\right] = \alpha e^{\alpha t}V(t) + e^{\alpha t}V'(t). \tag{3.2}$$

We substitute 3.1 in 3.2 and integrate over $(0,t)$ to obtain

$$\begin{aligned} e^{\alpha t}V(t) - V(0) = &\tfrac{\alpha}{2}\int_0^t e^{\alpha s}\int_\Omega \left(|u_t|^2 + |\nabla u|^2 + |\nabla u_t|^2\right)dxds \\ &-\alpha\varepsilon\int_0^t e^{\alpha s}\int_\Omega u_t u dxds - \alpha\varepsilon\int_0^t e^{\alpha s}\int_\Omega \nabla u\nabla u_t dxds + \alpha\int_0^t e^{\alpha s}\Psi(s)ds \\ &+\int_0^t e^{\alpha s}\int_\Omega \nabla u_t\int_0^s h(s-z)\nabla u(z)dzdxds - \varepsilon\int_0^t e^{\alpha s}\int_\Omega |u_t|^2\,dxds \\ &-\varepsilon\int_0^t e^{\alpha s}\int_\Omega |\nabla u_t|^2\,dxds + \varepsilon\int_0^t e^{\alpha s}\int_\Omega |\nabla u|^2\,dxds \\ &-\varepsilon\int_0^t e^{\alpha s}\int_\Omega \nabla u\int_0^s h(s-z)\nabla u(z)dzdxds + \int_0^t \Psi'(s)e^{\alpha s}ds. \end{aligned}$$

Equivalently,

$$
\begin{aligned}
e^{\alpha t}V(t) &= V(0) + (\tfrac{\alpha}{2}-\varepsilon)\int_0^t\int_\Omega e^{\alpha s}u_t^2\,dxds + (\tfrac{\alpha}{2}+\varepsilon)\int_0^t\int_\Omega e^{\alpha t}\left|\nabla u\right|^2 dxds \\
&+(\tfrac{\alpha}{2}-\varepsilon)\int_0^t\int_\Omega e^{\alpha t}\left|\nabla u_t\right|^2 dxds + \alpha\int_0^t e^{\alpha s}\Psi(s)dx - \alpha\varepsilon\int_0^t\int_\Omega e^{\alpha s}u_t u\,dxds \\
&-\alpha\varepsilon\int_0^t\int_\Omega e^{\alpha s}\nabla u\nabla u_t\,dxds + \int_0^t\int_\Omega e^{\alpha s}\nabla u_t\int_0^s h(s-z)\nabla u(z)\,dzdxds \\
&-\varepsilon\int_0^t\int_\Omega e^{\alpha s}\nabla u\int_0^s h(s-z)\nabla u(z)\,dzdxds + \int_0^t e^{\alpha s}\Psi'(s)ds.
\end{aligned}
\tag{3.3}
$$

By Lemma 2.2 (with $\delta = 1/2\alpha$ and $\delta = 1/4\alpha$, respectively) and Poincaré inequality, we have

$$
\int_0^t\int_\Omega u_t u\,dxds \le \frac{1}{2\alpha}\int_0^t\int_\Omega u_t^2\,dxds + \frac{\alpha C_p}{2}\int_0^t\int_\Omega \left|\nabla u\right|^2 dxds,
\tag{3.4}
$$

and

$$
\int_0^t\int_\Omega \nabla u_t\nabla u\,dxds \le \frac{1}{4\alpha}\int_0^t\int_\Omega \left|\nabla u_t\right|^2 + \alpha\int_0^t\int_\Omega \left|\nabla u\right|^2 dxds.
\tag{3.5}
$$

With the help of these two estimations we find

$$
\begin{aligned}
&e^{\alpha t}V(t) + \varepsilon\int_0^t\int_\Omega e^{\alpha s}\nabla u\int_0^s h(s-z)\nabla u(z)\,dzdxds \\
&\le V(0) + \tfrac{1}{2}(\varepsilon+\alpha-2\varepsilon)\int_0^t\int_\Omega e^{2\lambda s}u_t^2\,dxds + \alpha\int_0^t e^{\alpha s}\Psi(s)ds \\
&+\tfrac{1}{2}\left[\alpha^2\varepsilon(C_p+2)+\alpha+2\varepsilon\right]\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2 dxds \\
&+\tfrac{1}{4}(2\alpha-3\varepsilon)\int_0^t\int_\Omega e^{2\lambda s}\left|\nabla u_t\right|^2 dxds + \int_0^t e^{\alpha s}\Psi'(s)ds \\
&+\int_0^t\int_\Omega e^{\alpha s}\nabla u_t\int_0^s h(s-z)\nabla u(z)\,dzdxds.
\end{aligned}
$$

A differentiation of $\Psi(t)$ with respect to t gives

$$
\begin{aligned}
\Psi'(t) &= \bar{h}_\alpha\int_\Omega \left(\eta\left|\nabla u\right|^2 + \mu\left|\nabla u_t\right|^2\right)dx \\
&-\int_\Omega\int_0^t h(t-s)\left(\eta\left|\nabla u\right|^2 + \mu\left|\nabla u_t\right|^2\right)dsdx - \alpha\Psi(t).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^t e^{\alpha s}\Psi'(s)ds &= \bar{h}_\alpha\int_0^t e^{\alpha s}\int_\Omega \left(\eta\left|\nabla u\right|^2 + \mu\left|\nabla u_t\right|^2\right)dxds \\
-\int_0^t e^{\alpha s}\int_\Omega\int_0^s h(s-z)&\left(\eta\left|\nabla u(z)\right|^2 + \mu\left|\nabla u_t(z)\right|^2\right)dzdxds - \alpha\int_0^t e^{\alpha s}\Psi(s)ds.
\end{aligned}
\tag{3.6}
$$

Now, by Lemma 2.2 (with $\delta = \varepsilon/16$), we can write

$$\int_0^t\int_\Omega e^{\alpha s}\nabla u_t\int_0^s h(s-z)\nabla u(z)dzdxds$$
$$=\int_0^t\int_\Omega e^{\frac{\alpha}{2}s}\nabla u_t\int_0^s h(s-z)e^{\frac{\alpha}{2}(s-z)}e^{\frac{\alpha}{2}z}\nabla u(z)dzdxds$$
$$\leq\frac{\varepsilon}{16}\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2dxds+\frac{4}{\varepsilon}\int_0^t\int_\Omega\left|\int_0^s h(s-z)e^{\frac{\alpha}{2}(s-z)}e^{\frac{\alpha}{2}z}\nabla u(z)dz\right|^2dxds.$$

But

$$\left|\int_0^s h(s-z)e^{\frac{\alpha}{2}(s-z)}e^{\frac{\alpha}{2}z}\nabla udz\right|^2\leq\bar{h}_{\alpha/2}\int_0^s h(s-z)e^{\frac{\alpha}{2}(s-z)}e^{\alpha z}\left|\nabla u(z)\right|^2dz$$
$$\leq\bar{h}_{\alpha/2}e^{\alpha s}\int_0^s h(s-z)\left|\nabla u(z)\right|^2dz.$$

Therefore,

(3.7)
$$\int_0^t\int_\Omega e^{\alpha s}\nabla u_t\int_0^s h(s-z)\nabla u(z)dzdxds\leq\frac{\varepsilon}{16}\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2dxds$$
$$+\frac{4\bar{h}_{\alpha/2}}{\varepsilon}\int_0^t e^{\alpha s}\int_\Omega\int_0^s h(s-z)\left|\nabla u(z)\right|^2dzdxds.$$

We also have by the definition of the quadratic term $Q(w,t,k)$, that

(3.8)
$$\int_0^t\int_\Omega e^{\alpha s}\nabla u\int_0^s h(s-z)\nabla u(z)dzdxds$$
$$=\int_0^t\int_\Omega e^{\frac{\alpha}{2}s}\nabla u\int_0^s h(s-z)e^{\frac{\alpha}{2}(s-z)}e^{\frac{\alpha}{2}z}\nabla u(z)dzdxds$$
$$=Q\left(e^{\frac{\alpha}{2}t}\nabla u,t,h(t)e^{\frac{\alpha}{2}t}\right)\geq 0.$$

Taking into account 3.4 - 3.8 in 3.3 we obtain

(3.9)
$$e^{\alpha t}V(t)+\varepsilon Q\left(e^{\frac{\alpha}{2}t}\nabla u,t,h(t)e^{\frac{\alpha}{2}t}\right)\leq V(0)+\frac{1}{2}(\varepsilon+\alpha-2\varepsilon)\int_0^t\int_\Omega e^{\alpha s}u_t^2dxds$$
$$+\frac{1}{16}(8\alpha+16\mu\bar{h}_\alpha-11\varepsilon)\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2dxds$$
$$+\frac{1}{2}\left[\alpha^2\varepsilon(C_p+2)+\alpha+2\varepsilon+2\eta\bar{h}_\alpha\right]\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2dxds$$
$$+\left[\frac{4}{\varepsilon}\int_0^\infty h(s)e^{\frac{\alpha}{2}s}ds-\eta\right]\int_0^t\int_\Omega e^{\alpha s}\int_0^s h(s-z)\left|\nabla u(z)\right|^2dzdxds$$
$$-\mu\int_0^t\int_\Omega e^{\alpha s}\int_0^s h(s-z)\left|\nabla u_t(z)\right|^2dzdxds.$$

Let us add $\varepsilon Q\left(\left(e^{\frac{\alpha}{2}t}\nabla u\right)_t,t,h(t)e^{\frac{\alpha}{2}t}\right)$ to both sides of 3.9 and make use of the following estimation obtained with the help of Lemma 2.2 (with $\delta=1/2$) and the inequality $(a+b)^2\leq 2a^2+2b^2$,

$$Q\left(h(t)e^{\frac{\alpha}{2}t},t,(e^{\frac{\alpha}{2}t}\nabla u)_t\right)\leq\frac{\alpha^2}{8}\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2dxds$$
$$+\frac{1}{2}\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2dxds+\frac{\alpha^2}{2}\bar{h}_{\alpha/2}\int_0^t e^{\alpha s}\int_\Omega\int_0^s h(s-z)\left|\nabla u(z)\right|^2dzdxds$$
$$+\frac{1}{2}\bar{h}_{\alpha/2}\int_0^t e^{\alpha s}\int_\Omega\int_0^s h(s-z)\left|\nabla u_t(z)\right|^2dzdxds$$

we find

$$
\begin{aligned}
e^{\alpha t}V(t) + \varepsilon \left[Q\left(h(t)e^{\frac{\alpha}{2}t}, t, e^{\frac{\alpha}{2}t}\nabla u\right) + Q\left(h(t)e^{\frac{\alpha}{2}t}, t, (e^{\frac{\alpha}{2}t}\nabla u)_t\right)\right] \\
\leq V(0) + \tfrac{1}{2}(\varepsilon + \alpha - 2\varepsilon)\int_0^t\int_\Omega e^{\alpha s}u_t^2 dxds \\
+\tfrac{1}{16}(8\alpha + 16\mu\bar{h}_\alpha - 3\varepsilon)\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2 dxds \\
+\tfrac{1}{2}\left[\tfrac{\alpha^2\varepsilon}{4}(4C_p+9) + \alpha + 2\varepsilon + 2\eta\bar{h}_\alpha\right]\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2 dxds \\
+\left[\tfrac{4\bar{h}_{\alpha/2}}{\varepsilon} - \eta + \tfrac{\varepsilon\alpha^2\bar{h}_{\alpha/2}}{2}\right]\int_0^t\int_\Omega e^{\alpha s}\int_0^s h(s-z)\left|\nabla u(z)\right|^2 dzdxds \\
-\left(\mu - \tfrac{\varepsilon\bar{h}_{\alpha/2}}{2}\right)\int_0^t\int_\Omega e^{\alpha s}\int_0^s h(s-z)\left|\nabla u_t(z)\right|^2 dzdxds.
\end{aligned}
$$

Taking $\mu = \frac{\varepsilon\bar{h}_{\alpha/2}}{2}$ and $\eta = \left(\frac{4}{\varepsilon} + \frac{\varepsilon\alpha^2}{2}\right)\bar{h}_{\alpha/2}$, this last relation reduces to

$$
\begin{aligned}
e^{\alpha t}V(t) + \varepsilon \left[Q\left(h(t)e^{\frac{\alpha}{2}t}, t, e^{\frac{\alpha}{2}t}\nabla u\right) + Q\left(h(t)e^{\frac{\alpha}{2}t}, t, (e^{\frac{\alpha}{2}t}\nabla u)_t\right)\right] \\
\leq V(0) + \tfrac{1}{2}(\varepsilon + \alpha - 2\varepsilon)\int_0^t\int_\Omega e^{\alpha s}u_t^2 dxds \\
+\tfrac{1}{16}(8\alpha + 16\mu\bar{h}_\alpha - 3\varepsilon)\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2 dxds \\
+\tfrac{1}{2}\left[\tfrac{\alpha^2\varepsilon}{4}(4C_p+9) + \alpha + 2\varepsilon + 2\eta\bar{h}_\alpha\right]\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2 dxds.
\end{aligned}
\tag{3.10}
$$

Next, in view of Lemma 2.3, we infer from 3.10 that

$$
\begin{aligned}
e^{\alpha t}V(t) + \left\{\varepsilon L - \tfrac{1}{2}\left[\tfrac{\alpha^2\varepsilon}{4}(4C_p+9) + \alpha + 2\varepsilon + 2\eta\bar{h}_\alpha\right]\right\}\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u\right|^2 dxds \\
\leq V(0) + \tfrac{1}{16}(8\alpha + 16\mu\bar{h}_\alpha - 3\varepsilon)\int_0^t\int_\Omega e^{\alpha s}\left|\nabla u_t\right|^2 dxds \\
+\varepsilon L\int_\Omega \left|\nabla u_0\right|^2 dxds + \tfrac{1}{2}(\varepsilon + \alpha - 2\varepsilon)\int_0^t\int_\Omega e^{\alpha s}u_t^2 dxds
\end{aligned}
$$

with $L \geq 2$. If $\alpha \leq \varepsilon/8$, $\bar{h}_\alpha \leq \varepsilon/16$ and $\varepsilon < 1/(1 + C_p)$, then we clearly end up with

$$
e^{\alpha t}V(t) \leq V(0) + \varepsilon L\int_\Omega \left|\nabla u_0\right|^2 dxds. \tag{3.11}
$$

The conclusion follows from 3.11 and Proposition 2.1. ∎

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