On the Wave Equation with a Dissipation and a Source of Cubic Convolution Type in $\mathbb{R}^N$

Nasser-eddine Tatar
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran, 31261 Saudi Arabia
E-mail: tatarn@kfupm.edu.sa

Abstract

We study the interaction between a dissipative term and a source term of cubic convolution type for the wave equation in $\mathbb{R}^N$. These terms have both the same form and involve convolutions with a singular kernel. The investigation will depend on the coefficient of the source term which is a function of the time variable. Some results on the boundedness of the solutions are proved. Moreover, we establish an asymptotic stability result.

AMS Subject Classification: 35B35, 35B40, 35L15, 35L70.

Key Words: Asymptotic behavior, cubic convolution, nonlinear dissipation, polynomial growth, singular kernel.

1 Introduction

We consider the following problem

$$\begin{cases} u_{tt} + mu + \mu u_t(V_\gamma * u_t^2) = \Delta u + \lambda h(t) u(V_\gamma * u^2), \text{ in } (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x \in \mathbb{R}^N \end{cases} \quad (1)$$

where $V_\gamma = |x|^{-\gamma}$, $(V_\gamma * w)(x) = \int_{\mathbb{R}^N} V_\gamma(x-y)w(y)dy$, $0 < \gamma < N$ and m, μ, $\lambda \geq 0$. The potential $h(t) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and Δ stands for the Laplacian in $\mathbb{R}^N$.

In case $\mu = 0$ and $h(t)$ is constant, some global existence, scattering (existence of asymptotically free solutions) and global non existence results are shown in Perla Menzala and Strauss [9]. In fact, a blow up result has been proved but only for large enough initial data. For small initial data we cite the result of Hidano [3] established in case $N = 3$ and provided that the initial data are in addition spherically symmetric.

When $\lambda = 0$, some global existence results may be found in [5,8,10]. As for the asymptotic behavior we mention the work of Mochizuki and Motai in [7]. There the authors, using a weighted energy norm, obtained some results on the decay of the classical energy and determined its rate of convergence. They also established a non-decay result. It has been proved that one can find initial data for which the energy never decays at infinity.

In the present work we consider the case where both μ and λ are different from zero ($\mu \neq 0$, $\lambda \neq 0$) and a non constant potential $h(t)$. Using more or less standard arguments one can prove several existence theorems. These results will depend on the space dimension N, the parameters m ($m > 0$ or $m = 0$) and γ, and the regularity of the initial data (u_0, u_1). In particular, we have global existence of a solution in $C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ in case $(u_0, u_1) \in H^1 \times L^2$ with some restriction on N (if $m = 0$ we consider initial data with compact support). The restriction on N may be relaxed by assuming more regular initial data (for instance $(u_0, u_1) \in H^2 \times \left(H^1 \cap L^{6N/(3N-2\gamma)}\right)$). Here we shall not be concerned about these restrictions as our results are valid for all $(u_0, u_1) \in H^1 \times L^2$ (provided that we have existence) and some conditions only on γ. In this paper, our main interest is to investigate the balance between the dissipation and the source terms. To this end we shall follow closely the work of Georgiev and Milani [2] where a different problem was treated. Namely, they studied the wave equation in a bounded domain with dissipation and source terms of polynomial type ($u_t |u_t|^{p-1}$ and $u |u|^{p-1}$ respectively). We will rely on some of their estimates.

The deal with the difficulties generated by the unboundedness of the spatial region, the nonlocal nature of the dissipation and the source terms as well as the singularity of the kernel represents the feature of our present investigation.

As we shall see, we obtain essentially the same results as in [2]. Namely, when $h(t) = O((1+t)^{-\delta})$ as $t \to +\infty$ and $0 \leq \delta \leq 3$ there are initial data yielding polynomial growth of the energy as $t \to +\infty$. If $\delta > 3$, the energy is uniformly bounded for any choice of the initial values. In addition, an asymptotic stability result is proved. The power $\delta = 3$ is then the critical exponent. Roughly, this means that the cubic convolutions behave as power nonlinearities of order 3. Finally, we mention here that in case of a polynomial source of the form $u\,|u|^{p-1}$, with $p \geq 5$ the author proved in [11] a blow up result in finite time.

2 Polynomial growth

In this section we state and prove our first result. It asserts that we can find initial data for which the corresponding energy goes to infinity as $t \to +\infty$, provided that the potential is weakly decaying. The rate of growth is polynomial. Although we assume $m > 0$, the results remain valid (under additional hypotheses) for the case $m = 0$ (see the Remarks 3 and 4 below).

Lemma 1. *(Hardy-Littlewood-Sobolev inequality, see [4] or [6])*
Let $u \in L^p(\mathbb{R}^N)$ $(p > 1)$, $0 < \gamma < N$ and $\frac{\gamma}{N} > 1 - \frac{1}{p}$, then $(1/\,|x|^\gamma) * u \in L^q(\mathbb{R}^N)$ with $\frac{1}{q} = \frac{\gamma}{N} + \frac{1}{p} - 1$. Also the mapping from $u \in L^p(\mathbb{R}^N)$ into $(1/\,|x|^\gamma) * u \in L^q(\mathbb{R}^N)$ is continuous.

Theorem 1. *Let $m > 0$ and $0 < \gamma \leq 4$. Suppose that $h(t)$ is not constant and is such that*

$$-\alpha h(t)^{\frac{4}{3}} \leq h'(t) \leq 0,\ \ \alpha > 0. \tag{2}$$

Then, there exists $\alpha_0 > 0$ and $\{u_0, u_1\} \in H^1 \times L^2$ such that if $\alpha \leq \alpha_0$, the solution of (1) grows polynomially as $t \to +\infty$.

Proof. A multiplication of the equation (1) by u_t and an integration over $\mathbb{R}^N$ yield

$$\begin{aligned}\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N}\left(u_t^2 + mu^2 + |\nabla u|^2\right)dx\right\} + \mu\int_{\mathbb{R}^N} u_t^2(V_\gamma * u_t^2)dx \\ = \lambda h(t)\int_{\mathbb{R}^N} u_t u(V_\gamma * u^2)dx.\end{aligned} \tag{3}$$

We denote by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^N}\left(u_t^2 + mu^2 + |\nabla u|^2\right)dx$$

the classical energy and by

$$E_\psi(t) = E(t) - \psi(t)\int_{\mathbb{R}^N} u_t u\,dx$$

the modified energy. Here $\psi(t)$ is a C^1-function to be precised later. Clearly,

$$\frac{dE_\psi(t)}{dt} = \frac{dE(t)}{dt} - \psi^{'}(t)\int\limits_{\mathbb{R}^N} u_t u dx - \psi(t)\int\limits_{\mathbb{R}^N} u_t^2 dx - \psi(t)\int\limits_{\mathbb{R}^N} u_{tt} u dx \tag{4}$$

and multiplying the equation in (1) by u we get

$$\begin{aligned}\int\limits_{\mathbb{R}^N} u_{tt} u dx = -\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx - m\int\limits_{\mathbb{R}^N} u^2 dx - \mu\int\limits_{\mathbb{R}^N} u_t u (V_\gamma * u_t^2) dx \\ +\lambda h(t)\int\limits_{\mathbb{R}^N} u^2 (V_\gamma * u^2) dx.\end{aligned} \tag{5}$$

Inserting (5) and (3) into (4) we find

$$\begin{aligned}&\tfrac{dE_\psi(t)}{dt} + \mu\int\limits_{\mathbb{R}^N} u_t^2 (V_\gamma * u_t^2) dx + \psi(t)\int\limits_{\mathbb{R}^N} u_t^2 dx + \lambda\psi(t)h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx \\ &= \lambda h(t)\int\limits_{\mathbb{R}^N} u_t u (V_\gamma * u^2) dx + \mu\psi(t)\int\limits_{\mathbb{R}^N} u_t u (V_\gamma * u_t^2) dx + \psi(t)\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\qquad + m\psi(t)\int\limits_{\mathbb{R}^N} u^2 dx - \psi^{'}(t)\int\limits_{\mathbb{R}^N} u_t u dx.\end{aligned} \tag{6}$$

This will be our reference equality for this proof and the next ones. Note that $h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx$ is a C^1-function and

$$\begin{aligned}&\tfrac{d}{dt}\left(h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx\right) \\ &= h^{'}(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx + 2h(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u^2)dx + 2h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u_t u)dx \\ &= h^{'}(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx + 4h(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u^2)dx.\end{aligned}$$

Therefore (5) may be written as

$$\begin{aligned}&\tfrac{d}{dt}\left(E_\psi(t) - \tfrac{\lambda}{4}h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx\right) + \mu\int\limits_{\mathbb{R}^N} u_t^2(V_\gamma * u_t^2)dx \\ &\qquad + \lambda\psi(t)h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx + \psi(t)\int\limits_{\mathbb{R}^N} u_t^2 dx \\ &= \mu\psi(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u_t^2)dx + \psi(t)\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + m\psi(t)\int\limits_{\mathbb{R}^N} u^2 dx \\ &\qquad - \tfrac{\lambda}{4}h^{'}(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx - \psi^{'}(t)\int\limits_{\mathbb{R}^N} u_t u dx.\end{aligned} \tag{7}$$

By the Parseval equality, the Cauchy-Schwarz inequality and the convolution property enjoyed by the kernel V_γ (see [4, Chapter 7.1 and 3.4]), we have

$$\begin{aligned}\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u_t^2)dx &\le \left[\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * (u_t u)\right)^2 dx\right]^{\frac{1}{2}} \\ &\le \left[\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{\frac{3}{4}} \left[\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx\right]^{\frac{1}{4}}.\end{aligned} \tag{8}$$

Now Young's inequality allows us to write

$$\psi(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u_t^2)dx \le \frac{3}{4}\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx + \frac{\psi^4(t)}{4}\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx. \tag{9}$$

We put

$$F(t) = E_\psi(t) - \frac{\lambda}{4}h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx.$$

From (7), (9) and the equality

$$\int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u\right)^2 dx = \int\limits_{\mathbb{R}^N} u(V_\gamma * u)dx$$

which follows from the argument

$$\begin{aligned}\int\limits_{\mathbb{R}^N} u(V_\gamma * u)dx &= \int\limits_{\mathbb{R}^N} \widehat{u}\, \overline{\widehat{V_\gamma * u}}dx \\ = \int\limits_{\mathbb{R}^N} \widehat{u}\left|\widehat{V_{\frac{N+\gamma}{2}}}\right|^2 \overline{\widehat{u}}dx &= \int\limits_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u\right)^2 dx,\end{aligned}$$

we get

$$\begin{aligned}\tfrac{dF(t)}{dt} + \tfrac{1}{4}\mu \int\limits_{\mathbb{R}^N} u_t^2(V_\gamma * u_t^2)dx + \psi(t)\int\limits_{\mathbb{R}^N} u_t^2 dx + \psi'(t)\int\limits_{\mathbb{R}^N} u_t u dx \\ + \left\{\lambda\psi(t)h(t) + \tfrac{\lambda}{4}h'(t) - \mu\tfrac{\psi^4(t)}{4}\right\}\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx \\ \le \psi(t)\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + m\psi(t)\int\limits_{\mathbb{R}^N} u^2 dx.\end{aligned}$$

We may write

$$\begin{aligned}\tfrac{dF(t)}{dt} &\le k(t)F(t) - \left(\tfrac{k(t)}{2} + \psi(t)\right)\int\limits_{\mathbb{R}^N} u_t^2 dx + \left(\psi(t) - \tfrac{k(t)}{2}\right)\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx \\ &+ m\left(\psi(t) - \tfrac{k(t)}{2}\right)\int\limits_{\mathbb{R}^N} u^2 dx + \left(k(t)\psi(t) - \psi'(t)\right)\int\limits_{\mathbb{R}^N} u_t u dx \\ &- \left\{\lambda\psi(t)h(t) + \tfrac{\lambda}{4}h'(t) - \mu\tfrac{\psi^4(t)}{4} - \tfrac{\lambda}{4}k(t)h(t)\right\}\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx\end{aligned} \tag{10}$$

for some function $k(t)$. Setting

$$\psi(t) = ah^{\frac{1}{3}}(t) \text{ and } k(t) = bh^{\frac{1}{3}}(t),\ a < 1$$

with $2a < b < 4a$ and $\frac{4a-b}{a^4} > \frac{\mu}{\lambda}$ (this is satisfied if we choose, for instance, b such that $2a < b < 3a$ and a so that $a^3 < \lambda/\mu$), we get

$$\begin{aligned}\tfrac{dF(t)}{dt} &\le bh^{\frac{1}{3}}(t)F(t) - \left(\tfrac{b}{2} + a\right)h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} u_t^2 dx + \left(a - \tfrac{b}{2}\right)h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx \\ &+ m\left(a - \tfrac{b}{2}\right)\int\limits_{\mathbb{R}^N} u^2 dx + \tfrac{1}{2}\left|abh^{\frac{2}{3}}(t) - \tfrac{a}{3}h^{-\frac{2}{3}}(t)h'(t)\right|\left(\int\limits_{\mathbb{R}^N} u_t^2 dx + \int\limits_{\mathbb{R}^N} u^2 dx\right) \\ &- \tfrac{1}{4}\left\{(4\lambda a - \mu a^4 - \lambda b)\, h^{\frac{4}{3}}(t) + \lambda h'(t)\right\}\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx.\end{aligned}$$

Observe then that $\alpha_0 = 4\lambda a - \mu a^4 - \lambda b > 0$ so that if $\alpha \le \alpha_0$ the condition (2) on $h(t)$ implies that the coefficient of $\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx$ in (10) is negative. Moreover, choosing a sufficiently small so that

$$h^{\frac{1}{3}}(0) \le \min\left\{\frac{3(b+2a)}{a(3b+\alpha)}, \left(\frac{3m(b-2a)}{a(3b+\alpha)}\right)^{\frac{1}{2}}\right\},$$

(recall that $h(t) \le h(0)$) we obtain

$$\begin{aligned}\left|abh^{\frac{2}{3}}(t) - \tfrac{a}{3}h^{-\frac{2}{3}}(t)h'(t)\right| &= abh^{\frac{2}{3}}(t) - \tfrac{a}{3}h^{-\frac{2}{3}}(t)h'(t) \\ &\le a\left(b + \tfrac{\alpha}{3}\right)h^{\frac{2}{3}}(t) \le a\left(b + \tfrac{\alpha}{3}\right)h^{\frac{1}{3}}(t)h^{\frac{1}{3}}(0) \\ &\le (b-2a)h^{\frac{1}{3}}(t).\end{aligned}$$

Also,

$$\left|abh^{\frac{2}{3}}(t) - \frac{a}{3}h^{-\frac{2}{3}}(t)h'(t)\right| \le a\left(b + \frac{\alpha}{3}\right)h^{\frac{2}{3}}(0) \le m(b-2a).$$

The inequality (10) reduces to

$$\frac{dF(t)}{dt} \leq k(t)F(t). \tag{11}$$

Now letting $H(t) = -F(t)$, we see that $H^{'}(t) \geq k(t)H(t)$.

If $H(0) > 0$, that is

$$\frac{1}{2}\int\limits_{\mathbb{R}^N} \left(u_1^2 + mu_0^2 + |\nabla u_0|^2\right) dx - ah^{\frac{1}{3}}(0)\int\limits_{\mathbb{R}^N} u_0 u_1 dx - \frac{\lambda}{4}h(0)\int\limits_{\mathbb{R}^N} u_0^2(V_\gamma * u_0^2)dx < 0$$

then,

$$H(t) \geq H(0)\exp\left\{\int\limits_0^t k(s)ds\right\}.$$

Clearly from (2),

$$\begin{aligned}\int\limits_0^t k(s)ds = b\int\limits_0^t h^{\frac{1}{3}}(s)ds \geq 3bh^{\frac{1}{3}}(0)\int\limits_0^t \tfrac{ds}{3+\alpha h^{\frac{1}{3}}(0)s} \\ \geq \tfrac{3b}{\alpha}\left\{\ln\left(3 + \alpha h^{\frac{1}{3}}(0)t\right) - \ln 3\right\} \\ \geq \tfrac{3b}{\alpha}\ln\left(1 + \tfrac{\alpha}{3}h^{\frac{1}{3}}(0)t\right).\end{aligned}$$

Hence,

$$H(t) \geq H(0)\left(1 + \frac{\alpha}{3}h^{\frac{1}{3}}(0)t\right)^{\frac{3b}{\alpha}}.$$

On the other hand,

$$\begin{aligned}H(t) = \tfrac{\lambda}{4}h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx + \psi(t)\int\limits_{\mathbb{R}^N} u_t u dx - \tfrac{1}{2}\int\limits_{\mathbb{R}^N}\left(u_t^2 + mu^2 + |\nabla u|^2\right)dx \\ \leq \tfrac{\lambda}{4}h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx + \tfrac{1}{2}(\psi^2(t) - m)\int\limits_{\mathbb{R}^N} u^2 dx.\end{aligned} \tag{12}$$

Clearly, by our previous assumption on a

$$a^2h^{\frac{2}{3}}(t) \leq ah^{\frac{2}{3}}(0) \leq \frac{m(b-2a)}{b+\frac{\alpha}{3}} \leq m$$

which implies that $\psi^2(t) - m \leq 0$. Consequently,

$$H(t) \leq \frac{\lambda}{4}h(t)\int\limits_{\mathbb{R}^N} u^2(V_\gamma * u^2)dx.$$

By (2) and the Hardy-Littlewood-Sobolev inequality (Lemma 1)

$$\begin{aligned} H(t) &\le \tfrac{\lambda}{4}h(0)\int_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx \\ &\le \tfrac{\lambda}{4}h(0)C\left\|u^2\right\|^2_{\frac{2N}{2N-\gamma}} \le \tfrac{\lambda}{4}h(0)C\left\|u\right\|^4_{\frac{4N}{2N-\gamma}}. \end{aligned} \tag{13}$$

Next, if $\gamma \le 4$ the Sobolev embedding $H^1 \subset L^q$ for $2 \le q \le \frac{2N}{N-2}$ if $N > 2$ and $q < \infty$ if $N \le 2$, implies

$$H(t) \le \tilde{C}\left\|u\right\|^4_{H^1} \tag{14}$$

for some positive constant $\tilde{C}$. We conclude that $\|u\|_{H^1}$ goes to infinity polynomially as $t \to +\infty$. □

Remark 1. *It is easy to see, using some Sobolev embeddings and (3) (see also [9]), that if the initial data $u_1(x)$ and $u_2(x)$ are of compact support, (say supp $u_1(x) \cup$ supp $u_2(x) \subset \{|x| < R\}$, for some $R > 0$), then the solution $u(x,t)$ is also of compact support (supp $u(t,.) \subset \{|x| < R+t\}$, for any $t < T_m$, where T_m is the maximal time of existence).*

Remark 2. *If the initial data are of compact support, then the following hold for the solutions of (1)*

(i) if $N = 1$, we have (by Poincaré inequality) $\|u\|_2 \le C(R+t)\|\nabla u\|_2$

(ii) if $N \ge 3$, we have (by [1, Theorem IX.9]) $\|u\|_{p^*} \le C\|\nabla u\|_2$, $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{N}$, therefore

$$\|u\|_2 \le C(R+t)\|u\|_{p^*} \le \tilde{C}(R+t)\|\nabla u\|_2$$

(iii) if $N = 2$, we have (by [1, Corollary IX.11), $\|u\|_2 \le C\|\nabla u\|_2$.

Remark 3. *The result of Theorem 2 is still valid in case $m = 0$ with initial data of compact support. To see this we use Remarks 1, 2 and an additional condition. Namely, we need the boundedness of $(R+t)h^{\frac{1}{3}}(t)$. This condition is not needed in case $N = 2$. Observe that this condition is not in contradiction with the assumption (2) on $h(t)$. Briefly, we shall point out here the main modifications to introduce in the proof of Theorem 2. Firstly, we adopt the estimates*

$$\begin{aligned} h^{\frac{2}{3}}(t)\int_{\mathbb{R}^N} u_t u dx &\le h^{\frac{1}{6}}(t)h^{\frac{1}{2}}(t)\int_{\mathbb{R}^N} u_t u dx \\ &\le \tfrac{1}{2}h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t^2 dx + \tfrac{1}{2}h(t)\int_{\mathbb{R}^N} u^2 dx \\ &\le \tfrac{1}{2}h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t^2 dx + \tfrac{1}{2}h(t)C^2(R+t)^2\int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned}$$

Then, we choose a small enough so that

$$a\left(b+\frac{\alpha}{3}\right)\le(b+2a) \text{ and } a\left(b+\frac{\alpha}{3}\right)h(t)C^2(R+t)^2\le(b-2a)h^{\frac{1}{3}}(t)$$

i.e.

$$a\left(b+\frac{\alpha}{3}\right)\le(b+2a) \text{ and } a\left(b+\frac{\alpha}{3}\right)C^2h^{\frac{2}{3}}(t)(R+t)^2\le(b-2a).$$

In this way, we may control $\left(k(t)\psi(t)-\psi'(t)\right)\int\limits_{\mathbb{R}^N}u_tudx$ *in (10) by the negative terms in* $\int\limits_{\mathbb{R}^N}u_t^2dx$ *and* $\int\limits_{\mathbb{R}^N}|\nabla u|^2dx$ *in the same inequality. We obtain (11).*

Secondly, in (12) we use the estimate

$$\begin{aligned}\psi(t)\int\limits_{\mathbb{R}^N}u_tudx&\le\tfrac{1}{2}\int\limits_{\mathbb{R}^N}u_t^2dx+\tfrac{\psi^2(t)}{2}\int\limits_{\mathbb{R}^N}u^2dx\\ &\le\tfrac{1}{2}\int\limits_{\mathbb{R}^N}u_t^2dx+\tfrac{1}{2}a^2C^2h^{\frac{2}{3}}(t)(R+t)^2\int\limits_{\mathbb{R}^N}|\nabla u|^2dx.\end{aligned}$$

Here again we need the boundedness of $(R+t)h^{\frac{1}{3}}(t)$ *and a should be small enough so that* $a^2C^2h^{\frac{2}{3}}(t)(R+t)^2\le1$.

3 Boundedness

In this section and the forthcoming one we take $\lambda=\mu$ for simplicity.

Theorem 2. *If* $m>0$ *and the potential* $h(t)$ *satisfies*

$$h'(t)\le-\alpha h^{\frac{4}{3}-\beta}(t)\le0,\ \alpha>0,\ 0<\beta<\frac{1}{3} \tag{15}$$

then for all such α, β *and initial data* $\{u_0,u_1\}\in H^1\times L^2$*, the classical energy is uniformly bounded.*

Proof. Recall the reference equality (6)

$$\begin{aligned}\tfrac{dE_\psi(t)}{dt}&+\lambda\int\limits_{\mathbb{R}^N}u_t^2(V_\gamma*u_t^2)dx+\psi(t)\int\limits_{\mathbb{R}^N}u_t^2dx+\lambda\psi(t)h(t)\int\limits_{\mathbb{R}^N}u^2(V_\gamma*u^2)dx\\ &=\lambda h(t)\int\limits_{\mathbb{R}^N}u_tu(V_\gamma*u^2)dx+\lambda\psi(t)\int\limits_{\mathbb{R}^N}u_tu(V_\gamma*u_t^2)dx\\ &+\psi(t)\int\limits_{\mathbb{R}^N}|\nabla u|^2dx+m\psi(t)\int\limits_{\mathbb{R}^N}u^2dx-\psi'(t)\int\limits_{\mathbb{R}^N}u_tudx.\end{aligned} \tag{16}$$

By a similar argument to that in (8) and (9) , we may consider the estimates

$$h(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u^2)dx \le \frac{3}{4}h^{\frac{4}{3}}(t)\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx + \frac{1}{4}\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx \tag{17}$$

and

$$\psi(t)\int\limits_{\mathbb{R}^N} u_t u(V_\gamma * u_t^2)dx \le \frac{\psi^4(t)}{4}\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx + \frac{3}{4}\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx. \tag{18}$$

Taking into account (17) and (18) in (16), we find

$$\begin{gathered}\tfrac{dE_\psi(t)}{dt} + h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} u_t^2 dx \\ \le \left\{\tfrac{3}{4}\lambda h^{\frac{4}{3}}(t) - \lambda\psi(t)h(t) + \tfrac{1}{4}\lambda\psi^4(t)\right\}\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dx \\ + \left(\tfrac{1}{4}\lambda + \tfrac{3}{4}\lambda - \lambda\right)\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx + \psi(t)\int\limits_{\mathbb{R}^N}|\nabla u|^2 dx \\ + m\psi(t)\int\limits_{\mathbb{R}^N} u^2 dx - \psi'(t)\int\limits_{\mathbb{R}^N} u_t u dx.\end{gathered}$$

Let us set $\psi(t) = h^{\frac{1}{3}}(t)$. We get

$$\begin{gathered}\tfrac{dE_\psi(t)}{dt} + h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} u_t^2 dx \\ \le h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N}|\nabla u|^2 dx + mh^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} u^2 dx - \tfrac{1}{3}h^{-\frac{2}{3}}(t)h'(t)\int\limits_{\mathbb{R}^N} u_t u dx \\ \le 2h^{\frac{1}{3}}(t)E(t) - \tfrac{1}{3\sqrt{m}}h^{-\frac{2}{3}}(t)h'(t)\int\limits_{\mathbb{R}^N} u_t.\sqrt{m}u dx.\end{gathered} \tag{19}$$

Recalling that $h'(t) \le 0$, we obtain

$$\begin{gathered}\tfrac{dE_\psi(t)}{dt} + h^{\frac{1}{3}}(t)\int\limits_{\mathbb{R}^N} u_t^2 dx \\ \le 2h^{\frac{1}{3}}(t)E(t) - \tfrac{1}{3\sqrt{m}}h^{-\frac{2}{3}}(t)h'(t)\left(\tfrac{1}{2}\int\limits_{\mathbb{R}^N} u_t^2 dx + \tfrac{m}{2}\int\limits_{\mathbb{R}^N} u^2 dx\right) \\ \le \left(2h^{\frac{1}{3}}(t) - \tfrac{1}{3\sqrt{m}}h^{-\frac{2}{3}}(t)h'(t)\right)E(t).\end{gathered} \tag{20}$$

On the other hand, as $h(t)$ is decreasing to zero we may choose $t_0 > 0$ such that

$$h^{\frac{2}{3}}(t) \le \frac{1}{4}m \text{ for } t \ge t_0.$$

Therefore, for $t \geq t_0$

$$\begin{aligned} \psi(t) \int_{\mathbb{R}^N} u_t u dx &\leq \tfrac{1}{4} \int_{\mathbb{R}^N} u_t^2 dx + h^{\frac{2}{3}}(t) \int_{\mathbb{R}^N} u^2 dx \\ &\leq \tfrac{1}{4} \int_{\mathbb{R}^N} u_t^2 dx + \tfrac{1}{4} m \int_{\mathbb{R}^N} u^2 dx \leq \tfrac{1}{2} E(t). \end{aligned} \tag{21}$$

Consequently,

$$\begin{aligned} E_\psi(t) &= \tfrac{1}{2} \int_{\mathbb{R}^N} \left(u_t^2 + mu^2 + |\nabla u|^2\right) dx - \psi(t) \int_{\mathbb{R}^N} u_t u dx \\ &\geq E(t) - \tfrac{1}{2} E(t) = \tfrac{1}{2} E(t). \end{aligned}$$

Integrating (20) over (t_0, t), it appears that

$$\frac{1}{2} E(t) \leq E_\psi(t_0) + \int_{t_0}^{t} \left(2h^{\frac{1}{3}}(s) - \frac{1}{3\sqrt{m}} h^{-\frac{2}{3}}(s) h'(s) \right) E(s) ds$$

i.e.

$$E(t) \leq 2E_\psi(t_0) + \int_{t_0}^{t} \left(4h^{\frac{1}{3}}(s) - \frac{2}{3\sqrt{m}} h^{-\frac{2}{3}}(s) h'(s) \right) E(s) ds.$$

Observe now that

$$\int_{t_0}^{t} h^{\frac{1}{3}}(s) ds \leq -\frac{1}{\alpha} \int_{t_0}^{t} h^{\beta - 1}(s) h'(s) ds \leq \frac{h^\beta(0)}{\alpha\beta}$$

and

$$-\int_{t_0}^{t} h^{-\frac{2}{3}}(s) h'(s) ds \leq 3h^{\frac{1}{3}}(0).$$

By Gronwall's inequality we obtain

$$E(t) \leq 2E_\psi(t_0) \exp\left\{ \frac{4h^\beta(0)}{\alpha\beta} + \frac{2h^{\frac{1}{3}}(0)}{\sqrt{m}} \right\}, \ t \geq t_0.$$

□

Remark 4. *When $m = 0$, we need in addition to (15) the boundedness of $(R+t)^2 h^{\frac{1}{3}-\gamma}(t)$ for some $0 < \gamma < \frac{1}{3}$. For (19), by the Remarks 1 and 2 we have*

$$\begin{aligned} -\tfrac{1}{3}h^{-\frac{2}{3}}(t)h'(t) \int_{\mathbb{R}^N} u_t u dx &\le -\tfrac{1}{3}h^{-\frac{2}{3}}(t)h'(t)\left(\tfrac{1}{2}\int_{\mathbb{R}^N} u_t^2 dx + \tfrac{1}{2}\int_{\mathbb{R}^N} u^2 dx\right) \\ &\le -h^{-\frac{2}{3}}(t)h'(t)C(R+t)^2E(t) \le -h^{-\frac{2}{3}}(t)h'(t)Ch^{\gamma-\frac{1}{3}}(t)E(t) \\ &\le -Ch^{\gamma-1}(t)h'(t) \end{aligned}$$

and clearly,

$$-\int_0^t h^{\gamma-1}(s)h'(s)ds \le \frac{1}{\gamma}h^{\gamma}(0) < \infty.$$

We use also the estimate

$$\begin{aligned} h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t u dx &\le h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t^2 dx + h^{\frac{1}{3}-\gamma}(t)h^{\gamma}(t)\int_{\mathbb{R}^N} u^2 dx \\ &\le h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t^2 dx + h^{\frac{1}{3}-\gamma}(t)h^{\gamma}(t)C^2(R+t)^2\int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\le h^{\frac{1}{3}}(t)\int_{\mathbb{R}^N} u_t^2 dx + Ch^{\gamma}(t)\int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned}$$

Choose $t_0 > 0$ such that $Ch^{\gamma}(t) \le \frac{1}{4}$, for all $t \ge t_0$. Assuming, without loss of generality, that $C \ge 1$ we see that $h^{\frac{1}{3}}(t) \le \frac{1}{4}$, $t \ge t_0$ and hence (21) is satisfied.

4 Asymptotic stability

Theorem 3. *Let $0 < \gamma \le 4$ and assume the same hypotheses as in Theorem 3. Then for all $\alpha > 0$, $0 < \beta < \frac{1}{3}$ and initial data $\{u_0, u_1\} \in H^1 \times L^2$ we have $\lim_{t\to+\infty} E(t) = 0$.*

Proof. Let us consider once more the reference equality (6). Taking $\psi(t) = -\rho(1+t)^{-\delta}$, for some ρ and $\delta > 0$ to be determined and using the estimates

(16) and (17), we find

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int\limits_{\mathbb{R}^N}\left(u_t^2+mu^2+|\nabla u|^2\right)dx+\rho(1+t)^{-\delta}\int\limits_{\mathbb{R}^N}u_t u dx\right\}\\&+\rho(1+t)^{-\delta}\int\limits_{\mathbb{R}^N}|\nabla u|^2\,dx+m\rho(1+t)^{-\delta}\int\limits_{\mathbb{R}^N}u^2dx\\&\le\left(\frac{3}{4}\lambda h^{\frac{4}{3}}(t)+\frac{1}{4}\lambda\rho^4(1+t)^{-4\delta}\right)\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}}*u^2\right)^2dx\\&+\rho(1+t)^{-\delta}\int\limits_{\mathbb{R}^N}u_t^2dx-\delta\rho(1+t)^{-\delta-1}\int\limits_{\mathbb{R}^N}u_t u dx\\&+\lambda\rho(1+t)^{-\delta}h(t)\int\limits_{\mathbb{R}^N}u^2(V_\gamma*u^2)dx.\end{aligned}$$

From the Cauchy-Schwarz inequality and the fact that

$$\int\limits_{\mathbb{R}^N}u^2(V_\gamma*u^2)dx=\int\limits_{\mathbb{R}^N}\left(V_{\frac{N+\gamma}{2}}*u^2\right)^2dx\le C\,\|u\|_{H^1}^4\le M$$

(by (13), (14) and Theorem 3) we obtain

$$\begin{aligned}\frac{dE_\psi(t)}{dt}+\rho(1+t)^{-\delta}E_\psi(t)&\le\frac{3}{2}\rho(1+t)^{-\delta}\int\limits_{\mathbb{R}^N}u_t^2dx+\rho^2(1+t)^{-2\delta}\int\limits_{\mathbb{R}^N}u_t u dx\\&+\left(\frac{3}{4}\lambda h^{\frac{4}{3}}(t)+\frac{1}{4}\lambda\rho^4(1+t)^{-4\delta}+\lambda\rho(1+t)^{-\delta}h(t)\right)M\\&-\delta\rho(1+t)^{-\delta-1}\int\limits_{\mathbb{R}^N}u_t u dx.\end{aligned}$$

Using again the previous result that the classical energy is uniformly bounded, we get

$$\begin{aligned}\frac{dE_\psi(t)}{dt}+\rho(1+t)^{-\delta}E_\psi(t)&\le a_1(1+t)^{-\delta}+a_2(1+t)^{-2\delta}+a_3h^{\frac{4}{3}}(t)\\&+a_4(1+t)^{-4\delta}+a_5(1+t)^{-\delta}h(t)\\&\le b_1(1+t)^{-\delta}+b_2h^{\frac{4}{3}}(t)+b_3(1+t)^{-\delta}h(t)\end{aligned}\tag{22}$$

for some positive constants a_i and b_i, $i=1,2,3$. If

$$\varphi(t)=-\int\limits_0^t\psi(s)ds=\rho\int\limits_0^t(1+s)^{-\delta}ds=\frac{\rho}{1-\delta}\left[(1+t)^{1-\delta}-1\right],$$

it appears from (22) that

$$\frac{d}{dt}\left(e^{\varphi(t)}E_\psi(t)\right)\le e^{\varphi(t)}\left\{b_1(1+t)^{-\delta}+b_2h^{\frac{4}{3}}(t)+b_3(1+t)^{-\delta}h(t)\right\}.$$

Next, an integration over $(0,t)$ yields

$$\begin{aligned} & e^{\varphi(t)} E_\psi(t) - E_\psi(0) \\ \leq & \int_0^t e^{\varphi(s)} \left\{ b_1(1+s)^{-\delta} + b_2 h^{\frac{4}{3}}(s) + b_3(1+s)^{-\delta} h(s) \right\} ds. \end{aligned}$$

Now by some estimates proved in [2] we have for sufficiently small values of δ

$$e^{-\varphi(t)} \int_0^t e^{\varphi(s)} \left\{ b_1(1+s)^{-\delta} + b_2 h^{\frac{4}{3}}(s) + b_3(1+s)^{-\delta} h(s) \right\} ds \to 0$$

as $t \to +\infty$.

On the other hand, if $\rho \leq \frac{\sqrt{m}}{2}$ then we can easily see that

$$E_\psi(t) \geq \frac{1}{2} E(t), \; t \geq 0.$$

Thus, $\lim_{t \to +\infty} E(t) = 0$. This completes the proof. □

Acknowledgment: The author would like to thank King Fahd University of Petroleum and Minerals, the Department of Mathematical Sciences for its support.

References

[1] H. Brezis, "Analyse Fonctionnelle: Théorie et Applications", Masson, Paris, New York, Barcelone, Milan, Mexico, Sao Paulo, 1983.

[2] V. Georgiev and A. Milani, *On the asymptotic behavior of semilinear wave equations with degenerate dissipation and source terms*, Nonlinear Diff. Eqs and Appl. Vol. 5 Issue 1 (1998), 53-68.

[3] K. Hidano, *Small data scattering and blow up for a wave equation with a cubic convolution*, Funkcialaj Ekvacioj, 43 (2000), 559-588.

[4] L. Hörmander, "The Analysis of Linear Partial Differential Operator I", Springer-Verlag, 1983.

[5] J. L. Lions and W. A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math. France, 93 (1965), 43-96.

[6] S. Mizohata, "The Theory of Partial Differential Equations", Cambridge University Press, 1973.

[7] K. Mochizuki and T. Motai, *On energy decay-nondecay problems for wave equations with nonlinear dissipative term in* $\mathbb{R}^N$, J. Math. soc. Japan, Vol. 47 No. 3 (1995), 405-421.

[8] T. Motai, *On the Cauchy problem for the nonlinear Klein-Gordon equation with a cubic convolution*, Tsukuba J. Math., 12 (1988), 353-369.

[9] G. Perla Menzala and W. A. Strauss, *On a wave equation with a cubic convolution*, J. Diff. Eqs. 43 (1982), 93-105.

[10] W. A. Strauss, The Energy Method in Nonlinear Partial Differential Equations, Brasil Inst. Math. Pure e aplicada, 1966.

[11] N.-e. Tatar, *Blow up for the wave equation with a nonlinear dissipation of cubic convolution type in* $\mathbb{R}^N$, Appl. Math. Comp., Vol. 148 Issue 3 (2004), 759-771.