On Proper Refinement of Bimatrix Games Extreme Nash Equilibria

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Abstract

The goal of this paper is to make automatic refinement of Bimatrix Game *Proper* extreme Nash equilibria possible. For that aim, we establish the definition of the *set of* ϵ*-proper equilibria* of a bimatrix game. We define a $0-1$ mixed quadratic program to generate a sequence of ϵ-proper extreme Nash equilibria. Regular refinement of the Nash equilibrium concept is used to conclude on the properness of some equilibria. Finally, we define antoher $0-1$ mixed quadratic program to identify non-proper extreme Nash equilibria.

Key-words : Enumeration, Refinement, Proper, Bimatrix Game, Extreme Nash Equilibrium, Perfect.

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1 Introduction

A *bimatrix game* is a strategic confrontation of two players I and II. A bimatrix game $G(A, B)$ is defined by a pair of $n \times m$ payoff matrices A and B. Each player has a finite number of actions to choose from. The deterministic choice of an action is called *pure strategy*. Player I has to choose between n pure strategies, while player II has to choose between m pure strategies.

Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called *mixed strategies*, and represented by probability vectors $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Hence, player I's payoff is $x_1^t A x_2$ and player II's payoff is $x_1^t B x_2$.

A Nash *equilibrium* is defined as a profile of strategies such that simultaneously player I maximizes his payoff given the strategic choice of player II and player II maximizes his payoff given the strategic choice of player I. An equilibrium point is a profile of strategies where neither player has an interest to unilaterally change his strategic choice unless the other player does so. A number of papers have already solved the problem of enumeration of all Bimatrix Game Nash extreme equilibria (see [5] and [3]).

When confronted to a situation where a huge number of equilibria can be considered to solve a game, decision makers would have to refine their choices using some other rational concepts in addition to the concept of Nash equilibrium. Perfect and *Proper* equilibria are two refinements of the concept of Nash equilibrium based on the idea that a reasonable equilibrium should be stable against slight perturbations in the equilibrium strategies.

Lack of analytical and numerical tools that can be used to generate such equilibria with robustness properties made these refinements rarely used in practice. This paper tries to answer the following question: How can we automatically detect *proper* extreme Nash equilibria ?

Section 1 recalls the definition of proper refinement concept and establishes the definition of the *set of ϵ-proper equilibria*. Section 2 proposes a mixed $0-1$ quadratic program in order to identify *non-proper* extreme Nash equilibria. *Essential*, *Quasi-strong*, *Isolated* and *Regular* refinement concepts are also used in order to conclude on the properness of some equilibria.

2 Set of ϵ-proper equilibria

The main idea behind the definition of the *proper* refinement of Nash equilibria is that a reasonable player would try harder to avoid important mistakes than he or she would try to avoid small ones. While any proper equilibrium profile is perfect, a perfect equilibrium profile could be non-proper.

Definition 2.1 *A bimatrix game profile (x_1^k, x_2^k) is said to be ϵ_k-proper equilibrium, for some $\epsilon_k > 0$, if the following conditions are satisfied:*

$$if \quad A_i x_2^k < A_h x_2^k, \quad then \quad x_{1i}^k \leq \epsilon_k x_{1h}^k, \quad \forall i, h \in \{1, 2, ..., n\}, \tag{2.1}$$

$$if \quad x_1^k B_j < x_1^k B_l, \quad then \quad x_{2j}^k \leq \epsilon_k x_{2l}^k, \quad \forall j, l \in \{1, 2, ..., m\}, \tag{2.2}$$

$$x_{1i}^k > 0, \quad \forall i \in \{1, 2, ..., n\}, \qquad x_{2j}^k > 0, \quad \forall j \in \{1, 2, ..., m\}. \tag{2.3}$$

To provide a practical tool to identify ϵ-proper equilibria and non-proper equilibria, for any $\epsilon \geq 0$ and $\sigma \geq 0$, we introduce the set

$$\begin{array}{lll}
\Omega_\epsilon^\sigma = \{ \ (x_1, x_2) \ : & \exists u, v \text{ such that} & \mathbb{1} x_1 = 1, \ \mathbb{1} x_2 = 1, \\
& \sigma \leq x_{1i}, & \forall i \in \{1, 2, ..., n\}, \\
& \sigma \leq x_{2j}, & \forall j \in \{1, 2, ..., m\}, \\
& & \\
& A_h x_2 \leq A_i x_2 + L u_{ih}, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, \\
& x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, \\
& u_{ih} + u_{hi} \leq 1, & \forall i, h \in \{1, 2, ..., n\}, \ i < h, \\
& u_{ih} \in \{0, 1\}, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, \\
& & \\
& x_1 B_l \leq x_1 B_j + L v_{jl}, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l, \\
& x_{2j} + v_{jl} \leq \epsilon x_{2l} + 1, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l, \\
& v_{jl} + v_{lj} \leq 1, & \forall j, l \in \{1, 2, ..., m\}, \ j < l, \\
& v_{jl} \in \{0, 1\}, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l \qquad \}.
\end{array}$$

The following proposition ensures that each element of Ω_ϵ^σ is an ϵ-proper equilibrium.

Proposition 2.2 *If a strategy profile $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for some $\epsilon > 0$ and $\sigma > 0$, then (x_1, x_2) is an ϵ-proper equilibrium.*

Proof. Suppose that (x_1, x_2) belongs to Ω_ϵ^σ, for some $\epsilon > 0$ and $\sigma > 0$. Let i and h be indices in $\{1, 2, ..., n\}$ such that $i \neq h$. Then the inequality $u_{ih} + u_{hi} \leq 1$ ensures that the combination $u_{ih} = 1$ and $u_{hi} = 1$ is not possible. Furthermore,

- if $u_{ih} = 0$ and $u_{hi} = 0$ then $A_h x_2 = A_i x_2$.
- if $u_{ih} = 1$, then $\epsilon x_{1i} \leq x_{1i} \leq \epsilon x_{1h} \leq x_{1h}$ implies that $x_{1h} \geq \epsilon x_{1i}$ and $u_{hi} = 0$, thus $A_i x_2 \leq A_h x_2$,

It follows that conditions (2.1) are satisfied. In a similar way, conditions (2.2) are satisfied using binary variables v_{jl}, for all $j, l \in \{1, 2, ..., m\}$ with $j \neq l$.

Finally, with $0 < \sigma \leq x_{2j}$, for all $j \in \{1, 2, ..., m\}$, the conditions (2.3) are satisfied. ∎

Conversely, the following proposition ensures that any ϵ-proper equilibrium belongs to Ω^σ_ϵ for all sufficiently small values of σ.

Proposition 2.3 *If a profile (x_1, x_2) is an ϵ-proper equilibrium for some $\epsilon > 0$, then there exists a $\bar{\sigma} > 0$ such that $(x_1, x_2) \in \Omega^\sigma_\epsilon$ for every $0 \leq \sigma \leq \bar{\sigma}$.*

Proof. If a profile (x_1, x_2) is an ϵ-proper equilibrium for some $\epsilon > 0$, conditions (2.1) can be reformulated using binary variables u_{ih}, for all $i, h \in \{1, 2, ..., n\}$, $i \neq h$:

$$\text{If } \begin{cases} A_i x_2 < A_h x_2, \\ x_{1i} \leq \epsilon x_{1h}, \end{cases} \text{ then } \begin{cases} A_i x_2 \leq A_h x_2 + L u_{hi}, \\ x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, \\ u_{hi} = 0, \\ u_{ih} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_h x_2 < A_i x_2, \\ x_{1h} \leq \epsilon x_{1i}, \end{cases} \text{ then } \begin{cases} A_h x_2 \leq A_i x_2 + L u_{ih}, \\ x_{1h} + u_{hi} \leq \epsilon x_{1i} + 1, \\ u_{ih} = 0, \\ u_{hi} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_i x_2 = A_h x_2, \\ x_{1i} \leq 1, \\ x_{1h} \leq 1, \end{cases} \text{ then } \begin{cases} A_i x_2 \leq A_h x_2 + L u_{hi}, & x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, \\ A_h x_2 \leq A_i x_2 + L u_{ih}, & x_{1h} + u_{hi} \leq \epsilon x_{1i} + 1, \\ u_{hi} = 0, & u_{ih} = 0. \end{cases}$$

In a similar way, conditions (2.2) can be reformulated using binary variables v_{jl}, for all $j, l \in \{1, 2, ..., m\}$, $j \neq l$.

And finally, conditions (2.3) ensure that there exists a $\bar{\sigma} > 0$, such that $\bar{\sigma} \leq x_{1i}$, for all $i \in \{1, 2, ..., n\}$ and $\bar{\sigma} \leq x_{2j}$, for all $j \in \{1, 2, ..., m\}$.

Then, for every σ such that $0 \leq \sigma \leq \bar{\sigma}$ and $\sigma > 0$:

$$\begin{aligned} \sigma \leq x_{1i}, &\quad \text{for all } i \in \{1, 2, ..., n\}, \\ \sigma \leq x_{2j}, &\quad \text{for all } j \in \{1, 2, ..., m\}. \end{aligned}$$

Thus, $(x_1, x_2) \in \Omega^\sigma_\epsilon$ for every σ, such that $0 \leq \sigma \leq \bar{\sigma}$ and $\sigma > 0$. ∎

Myerson [11] and Jansen [10] define a *proper* equilibrium to be the limit of an infinite sequence of ϵ_k-proper equilibria, with ϵ_k converging to zero.

Definition 2.4 *An equilibrium $(\hat{x}_1, \hat{x}_2)$ is said to be proper if there is a sequence of ϵ_k-proper equilibria (x_1^k, x_2^k) such that*

$$\lim_{k\to\infty} \epsilon_k = 0 \quad \textit{and} \quad \lim_{k\to\infty} (x_1^k, x_2^k) = (\hat{x}_1, \hat{x}_2). \tag{2.4}$$

The main difficulty in applying this definition is to find a convergent sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of positive real numbers making the sequence $\left\{(x_1^k, x_2^k)\right\}_{k\in\mathbb{N}}$ converge to $(\hat{x}_1, \hat{x}_2)$, where (x_1^k, x_2^k) are ϵ_k-proper for each $k \in \mathbb{N}$. However, since Myerson [11] showed that every bimatrix game posseses at least one proper equilibrium, we can be sure that such a sequence exists for every bimatrix game.

3 Detection of non-proper equilibria

In order to generate such sequence of positive real numbers, our idea is to define a family of parametrized mixed $0-1$ quadratic programs such that their solutions define a sequence of ϵ-proper equilibria, when the parameter σ converges to 0,.

Proposition 3.1 *The perfect equilibrium profile $(\hat{x}_1, \hat{x}_2)$ is a proper equilibrium if and only if the following $0-1$-mixed quadratic program is feasible for some $\bar{\sigma} > 0$, and if $\lim_{\sigma\to 0^+} f(\sigma) = 0$.*

$$\begin{aligned} f(\sigma) \quad = \quad & \min_{(x_1,x_2)\in\Omega_\epsilon^\sigma,\epsilon} && \epsilon \\ & s.t. && \hat{x}_{1i} - \epsilon \le x_{1i} \le \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1,2,...,n\}, \\ & && \hat{x}_{2j} - \epsilon \le x_{2j} \le \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1,2,...,m\}, \\ & && 0 \le \epsilon \le 1. \end{aligned} \tag{3.5}$$

Proof. Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma))$ be the optimal solution to (3.5) for some given perfect equilibrium profile $(\hat{x}_1, \hat{x}_2)$. Proposition (2.2) ensures that $(x_1(\sigma), x_2(\sigma))$, is an $\epsilon(\sigma)$-proper equilibrium. Conditions (2.4) were reformulated using the minimization of ϵ such that

$$\begin{aligned} &\hat{x}_{1i} - \epsilon \le x_{1i} \le \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1,2,...,n\}, \\ &\hat{x}_{2j} - \epsilon \le x_{2j} \le \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1,2,...,m\}, \end{aligned}$$

in order to make the ϵ-proper equilibrium converge to $(\hat{x}_1, \hat{x}_2)$.

Hence, if the mixed $0-1$ quadratic program (3.5) is feasible for all σ, when $\sigma > 0$ converges to 0, we can conclude from Proposition (2.3) that there is always an ϵ-proper equilibrium $(x_1, x_2) \in \Omega_\epsilon^\sigma$.

Moreover, if the perfect equilibrium $(\hat{x}_1, \hat{x}_2)$ is proper then the optimal objective function value $f(\sigma) = \epsilon(\sigma)$ should necessarily converge to 0, when $\sigma > 0$ converges to 0, to make the solution $(x_1(\sigma), x_2(\sigma))$ converge to $(\hat{x}_1, \hat{x}_2)$ at the same time. One can also notice that $f(0) = 0$.

Else, if $f(\sigma)$ does not converge to 0, when $\sigma > 0$ converges to 0, then such a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper does not exist, when ϵ converges to 0. In this case, we can prove that the equilibrium point is not proper.

∎

In conlusion, if $f(\sigma)$ converges to 0, when $\sigma > 0$ converges to 0, it is possible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$, when $\epsilon(\sigma)$ converges to 0.

We use this result by computing the value of $f(\sigma)$ for some small values of σ. The $0-1$-mixed quadratic program (3.5) is solved using the NEW-QP algorithm [14]. This algorithm is a new version of the QP algorithm [1]. The QP algorithm provides an ξ-optimal solution for feasible quadratic programs, where ξ is the precision parameter. In order to solve the $0-1$-mixed quadratic program (3.5) using NEW-QP, we have written the binary value constraints on the u and v variables using the quadratic constraints $u_{ih} - u_{ih}^2 = 0$ and $v_{jl} - v_{jl}^2 = 0$. Because of the discrete values taken by these binary variables, we can be sure that the NEW-QP algorithm provides the optimal solution to the mixed $0-1$ quadratic program (3.5). In some cases, the numerical noise which might appear makes it difficult to conclude numerically that an equilibrium is proper.

Corollary 3.2 *Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma))$ be an optimal solution to (3.5) for some $\sigma > 0$. Then $(x_1(\sigma), x_2(\sigma))$ is an $\epsilon(\sigma)$-proper equilibrium, and if $\sigma" > \sigma' > 0$, then $\epsilon(\sigma") \geq \epsilon(\sigma') \geq 0$.*

Proof. If $\sigma" > \sigma' > 0$, the $0-1$ mixed quadratic program (3.5) for $\sigma' > 0$ is a relaxation of $0-1$ mixed quadratic program (3.5) for $\sigma'' > 0$. In fact the only difference between these two programs is in the constraints of $\Omega_\epsilon^{\sigma'}$ and $\Omega_\epsilon^{\sigma''}$:

$$\begin{array}{ll} \sigma' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma' \leq x_{2j}, & \forall j \in \{1,2,...,m\}, \end{array} \quad \text{and} \quad \begin{array}{ll} \sigma'' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma'' \leq x_{2j}, & \forall j \in \{1,2,...,m\}. \end{array}$$

$$\Rightarrow \begin{array}{ll} \sigma' < \sigma'' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma' < \sigma'' \leq x_{2j}, & \forall j \in \{1,2,...,m\}. \end{array}$$

Thus, $\Omega_\epsilon^{\sigma''} \subseteq \Omega_\epsilon^{\sigma'}$ and $\epsilon(\sigma") \geq \epsilon(\sigma') \geq 0$. ∎

There are two possible outcomes when evaluating $f(\sigma)$ for some small values of σ. The first possibility is that $f(\sigma)$ appears to converge to zero. The second possibility is that $f(\sigma)$ appears to be bounded below by some strictly positive value, say $\bar{\epsilon}$.

3.1 Case 1: $f(\sigma)$ converges to zero

This numerical result is not enough to conclude that the equilibrium profile is proper. In this case we can use other refinements of the Nash equilibrium concept.

3.1.1 Essential equilibrium

According to Wu wen-tsun and Jiang Jia-he [15] the *Essential* refinement is based on the concept of stability of an equilibrium against slight perturbations in the payoffs of the game.

Definition 3.3 *A strategy profile (x_1, x_2) is an essential equilibrium of a bimatrix game $G(A, B)$ if there exists, with every neighborhood N_x of (x_1, x_2) a neighborhood N_G of (A, B) such that $G(A', B')$ has no equilibria in N_x for all $(A', B') \in N_G$.*

It is known that every essential equilibrium is perfect [6]. Jansen [8] paid special attention to equilibrium points that are *Quasi-strong* and *isolated* at the same time; these equilibria were found to be essential.

3.1.2 Quasi-strong equilibrium

For an equilibrium profile (x_1, x_2) of a bimatrix game $G(A, B)$, let $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$. Then $M(A, x_2)$ is defined as the set of pure best replies of player I against x_2:

$$M(A, x_2) = \{i \in N; e_i A x_2 = \max_{k \in N} e_k A x_2\}, \tag{3.6}$$

and similarly,

$$M(x_1, B) = \{j \in M; x_1 B e_j = \max_{k \in M} x_1 B e_j\}, \tag{3.7}$$

is the set of pure best replies of player II against x_1 (Harsanyi [7]).

The *carrier* of x_1, $C(x_1)$, is the set $\{i \in N; x_{1i} > 0\}$ and carrier of x_2, $C(x_2)$, is the set $\{j \in M; x_{2j} > 0\}$.

Definition 3.4 *Any equilibrium profile (x_1, x_2) is quasi-strong if*

$$C(x_1) = M(A, x_2) \text{ and } C(x_2) = M(x_1, B).$$

Jansen [8] showed that a quasi-strong and *isolated* equilibrium point is stable against slight perturbations of the payoffs of the game.

3.1.3 Isolated equilibrium

An equilibrium profile (x_1, x_2) of a bimatrix game $G(A, B)$ is said to be *isolated* if there exists a neighborhood N_x of (x_1, x_2) such that it is the only equilibrium of $G(A, B)$ in this neighborhood N_x. In other words, any isolated equilibrium is an extreme equilibrium defining an isolated maximal Nash subset. Enumeration of all maximal Nash subsets can be used in order to automatically detect isolated equilibria. Moreover, Jansen [8] proposed the following definition.

Definition 3.5 *Let (x_1, x_2) be a quasi-strong equilibrium of a bimatrix game $G(A, B)$ with $A, B > 0$. Then (x_1, x_2) is isolated if and only if $|C(x_1)| = |C(x_2)|$ and the matrices $[a_{ij}]_{i \in C(x_1), j \in C(x_2)}$ and $[b_{ij}]_{i \in C(x_1), j \in C(x_2)}$ are nonsingular.*

While this definition applies only for bimatrix games $G(A, B)$ such that $A, B > 0$, it is well known that every bimatrix game can be modified in order to make $A, B > 0$ and without changing the set of maximal Nash subsets. For example, this can easily be done by adding $1 + |a_{min}|$, with $a_{min} = min\ a_{ij}$, to each element of A and $1 + |b_{min}|$,with $b_{min} = min\ b_{ij}$, to each element of B.

Jansen [8] points out that an isolated equilibrium is essential if and only if it is quasi-strong. Moreover, van Damme [6] showed that an isolated and quasi-strong equilibrium point is perfect and proper. This was also obtained by Okada [13] for bimatrix games.

3.1.4 Regular equilibrium

For any *Regular* [8, 9] equilibrium we can conclude that it is proper. A *Regular* equilibrium profile was first defined by Harsanyi [7] such that the Jacobian of a mapping associated with the game evaluated at the equilibrium point is nonsingular. This definition was later improved by van Damme [6] for a two-person case. He proved that an equilibrium is regular if and only if it is *quasi-strong* and *isolated* and showed that such equilibria are *stongly stable* and proper.

Regular equilibria have all kind of robustness properties. Jansen [9] showed also that an equilibrium point of a bimatrix game is regular if and only if it is isolated and quasi-strong.

3.2 Case 2: $f(\sigma) \geq \bar{\epsilon}$

The case where $f(\sigma)$ appears to be bounded below by some strictly positive value $\bar{\epsilon}$ implies that there are no ϵ-proper equilibrium near $(\hat{x}_1, \hat{x}_2)$ for values of ϵ less than $\bar{\epsilon}$, and therefore $(\hat{x}_1, \hat{x}_2)$ would not be proper.

In (3.5), let us suppose that $f(\sigma)$ converges to $\bar{\epsilon} > 0$, when $\sigma > 0$ converges to 0. We define a $0-1$ mixed quadratic program with the same conditions as Ω, with $\epsilon \leq \bar{\epsilon}/2$ and maximizing σ. If the optimal objective function of this program is equal to *zero* we can conclude that it would be impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to this equilibrium. Therefore the equilibrium is not proper.

Theorem 3.6 *If the optimal objective value of the following* $0-1$ *mixed quadratic program*

$$\begin{array}{cll} \max\limits_{(x_1,x_2)\in\Omega_\epsilon^\sigma,\epsilon,\sigma} & \sigma & \\ s.t. & \hat{x}_{1i}-\epsilon \leq x_{1i} \leq \hat{x}_{1i}+\epsilon, & \forall i \in \{1,2,...,n\}, \\ & \hat{x}_{2j}-\epsilon \leq x_{2j} \leq \hat{x}_{2j}+\epsilon, & \forall j \in \{1,2,...,m\}, \\ & 0 \leq \epsilon \leq \bar{\epsilon}/2. & \end{array} \tag{3.8}$$

is zero for some $\bar{\epsilon} > 0$*, then the equilibrium* $(\hat{x}_1, \hat{x}_2)$ *is not proper.*

Proof. If the optimal objective value is equal to 0, it is impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$. The equilibrium $(\hat{x}_1, \hat{x}_2)$ is not proper. ∎

With this result, automatic detection of non-proper extreme Nash equilibria can be carried out over any set of extreme Nash equilibria of a bimatrix game.

The first example shows how the objective function does not converge to zero in the case of a non-perfect equilibrium.

Example 3.7 *Let* A *and* B *be the payoff matrices of a bimatrix game taken from Myerson [12]*

$$A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 3 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 0 \\ 0 & 2 \end{pmatrix}.$$

We have used the algorithms EχMIP [3, 4] and EEE [5] to enumerate all the extreme Nash equilibria of this bimatrix game. Both algorithms enmuerated five extreme Nash equilibria (Table 1).

As in [2] we have used a pair of linear programs to conclude on the perfectness of each equilibrium. Four extreme Nash equilibria are not perfect and therefore non-proper. As mentionned by Myerson [12], the first extreme Nash equilibrium is the

Table 1: Extreme Nash Equilibria for Myerson [12]

Eq.	x_1				x_2		α_1	α_2
1	0	0	1	0	1	0	6	6
2	0	1	0	0	0	1	4	4
3	0	1	0	0	1/3	2/3	4	4
4	1	0	0	0	0	1	4	4
5	1	0	0	0	1/3	2/3	4	4

only proper equilibrium of this game. While the optimal values of ϵ seem to converge to $\bar{\epsilon} = 0.618$, as σ approaches 0 (Figure 2) with non-perfect extreme equilibria 2, 3, 4 and 5. We define a $0-1$ mixed quadratic program with the same conditions as in (3.8), with $\epsilon \leq \bar{\epsilon}/2$ and maximizing σ. Such a $0-1$ mixed quadratic program has an optimal objective equal to zero. For the third extreme Nash equilibrium, Figure 2 shows how $f(\sigma) = \epsilon$ decreases when σ decreases from $min(\frac{1}{4}, \frac{1}{2}) = \frac{1}{4}$ to 0.

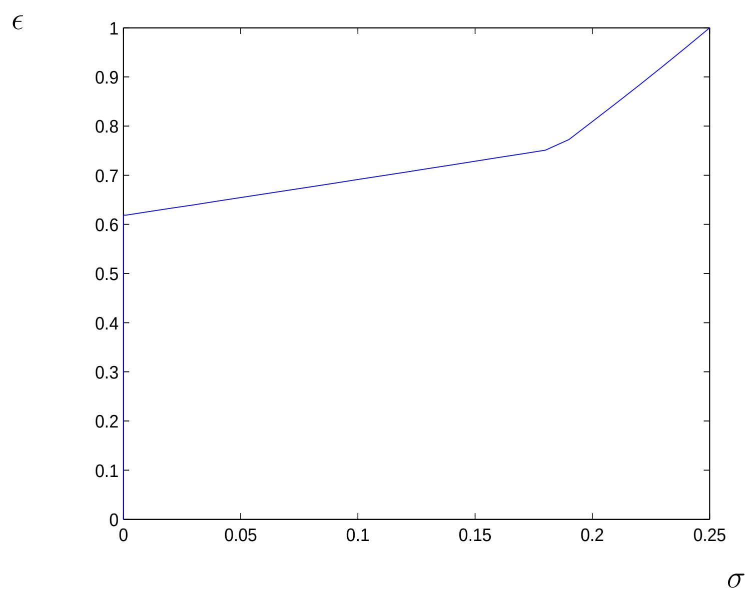

Figure 1: Plot of $\epsilon = \min f(\sigma)$

The set of extreme proper Nash equilibria defines the set of extreme points of all Maximal Myerson sets (Jansen [10]). There is only one maximal Myerson subset for the bimatrix game taken from Myerson [12].

The second example shows how some perfect equilibria can be non-proper. We also use Quasi-strong and isolated refinement concepts to conclude about the properness of some equilibria.

Example 3.8 *The following* (5×5) *bimatrix game has* 7 *extreme Nash equilibria identified in Table 2.*

$$A = B = \begin{pmatrix} 2 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 8 & 1 \\ 2 & 5 & 6 & 0 & 1 \\ 0 & 2 & 5 & 4 & 7 \\ 2 & 3 & 6 & 5 & 7 \end{pmatrix}$$

Table 2: Extreme Nash Equilibria for (5×5) bimatrix game

Eq.	x_1					x_2					α_1	α_2
1	0	0	0	0	1	0	0	0	0	1	7	7
2	0	0	0	1	0	0	0	0	0	1	7	7
3	0	0	1	0	0	0	0	1	0	0	6	6
4	0	0	1/6	0	5/6	0	0	1	0	0	6	6
5	0	1	0	0	0	0	0	0	1	0	8	8
6	1	0	0	0	0	0	0	0	0	1	7	7
7	7/8	1/8	0	0	0	0	0	0	3/4	1/4	25/4	25/4

This game has four maximal Nash subsets $T_1 = \{1, 2, 6\}$, $T_2 = \{3, 4\}$, $T_3 = \{5\}$ *and* $T_4 = \{7\}$. *Two regular extreme Nash equilibria are found.*

Equilibrium 5

$C(x_1) = \{2\}$*and* $C(x_2) = \{4\}$, $M(A, x_2) = \{2\}$ *and* $M(x_1, B) = \{4\} \Rightarrow$ *quasi-strong. The determinant of* $\begin{pmatrix} 8 \end{pmatrix}$ *is equal to* $8 \neq 0 \Rightarrow$ *isolated. This equilibrium is regular, essential, perfect and proper.*

Equilibrium 7

$C(x_1) = \{1, 2\}$*and* $C(x_2) = \{4, 5\}$, $M(A, x_2) = \{1, 2\}$ *and* $M(x_1, B) = \{4, 5\} \Rightarrow$ *quasi-strong. The determinant of* $\begin{pmatrix} 6 & 7 \\ 8 & 1 \end{pmatrix}$ *is equal to* $-50 \neq 0 \Rightarrow$ *isolated. This equilibrium is regular, essential, perfect and proper.*

4 Conclusion

In this paper we presented a mathematical programming approach for the refinement of extreme Nash equilibria. After complete enumeration of all extreme Nash equilibria, ϵ-proper extreme Nash equilibria are found using the convergence numerical

Table 3: Example (5×5)

Eq.	Perfect	Proper	$\bar{\epsilon}$	$\bar{\sigma}$	Quasi-strong	Isolated	Regular
1	yes	no	0.2894	5×10^{-3}	no	no	no
2	no	no	0.7325	10^{-3}	no	no	no
3	yes	yes	0.05627	10^{-5}	no	no	no
4	yes	no	0.2000	10^{-6}	yes	no	no
5	yes	yes	0.0564	10^{-5}	yes	yes	yes
6	yes	yes	0.054	10^{-5}	no	no	no
7	yes	yes	0.0776	6×10^{-5}	yes	yes	yes

results of a $0-1$ mixed quadratic program. Regular refinement helps concluding on the properness of some of these equilibria. Finally, non-proper extreme Nash equilibria are found using the result of another $0-1$ mixed quadratic program.

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