King Fahd University of Petroleum and Minerals
College of Sciences, Prep-Year Math Program

Code 001

Math 002, Exam II

(Term 012)
Saturday, April 27, 2002
6:30-8:10 p.m.

Code 001

Student's Name: KEY

ID #:__________ Section #:__________

This exam consists of Two parts

Part I: Multiple Choice: Encircle the correct answer.

Part II: Written Questions: Provide neat and complete solution. Show all necessary steps for full credit.

No Calculators, Pagers, or Mobiles are allowed during this exam.

Question	Points	Student's Score
Part I: (1 - 5)	20	
Part II:		
1	8	
2	8	
3	7	
4	7	
5	8	
6	8	
7	8	
8	10	
9	8	
10	8	

Total		
	100	

Part I: Multiple Choice Questions (4 Points for each correct answer)

Encircle the correct answer

Q. 1: **The period of** $f(x) = \sin 2x \cos 2x$ **is**

a) $\frac{\pi}{2}$

b) $\frac{\pi}{4}$

c) π^2

d) 2π

Q. 2: $\sin 80^\circ$ **is equal to**

a) $\sqrt{\frac{1+\cos 920^\circ}{2}}$

b) $\sqrt{\frac{1+\cos 880^\circ}{2}}$

c) $\sqrt{\frac{1-\cos 880^\circ}{2}}$

d) $\sqrt{\frac{1-\cos 40^\circ}{2}}$

Q 3. **Let** a **be a real number in** $[-1,1]$**. Then the equation**

$$\sin x = a, \quad \frac{\pi}{6} \le x < \frac{\pi}{6} + 4\pi$$

has

a) Exactly two solutions

b) Four solutions or two solutions

c) Two solutions or no solution

d) Four solutions or no solution

Q. 4: Which one of the following statements is true:

a) The function $f(x) = x + \sin x$ is periodic.

b) The amplitude of $y = \frac{3}{2}\cot(5x - \frac{\pi}{2}) + 2$ is $\frac{3}{2}$.

c) The graph of $y = \sec x$ has a maximum value when $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

d) The Equation $\arctan x = \frac{3\pi}{4}$ has a real solution.

Q. 5: The vertices of an ellipse with center at $(2,0)$ and major axis of length 6 on the x-axis are

a) $\left(\frac{1}{2},0\right),\left(3\frac{1}{2},0\right)$

b) $(-1,0),(5,0)$

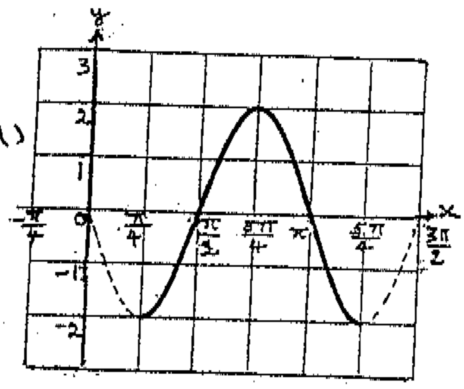
c) $(2,3),(2,-3)$

d) $\left(2,\frac{1}{2}\right),\left(2,-\frac{1}{2}\right)$

Part II: Written Questions

Q. 1: The figure given below represents the graph of a cosine function. Find

(8 points)



a) The amplitude

$m = -2, \ M = 2$

$\text{Amplitude} = \frac{M-m}{2} = \frac{2-(-2)}{2} = 2$ (1 pt)

b) The period

$\frac{5\pi}{4} - \frac{\pi}{4} = \pi$ (1 pt)

c) The phase shift

$\frac{\pi}{4}$ (1 pt)

d) The equation of the graph

$\frac{2\pi}{b} = \pi \quad \therefore b = 2$ (1 pt) $\quad -\frac{c}{b} = \frac{\pi}{4} \quad \therefore c = -\frac{\pi}{2}$ (1 pt)

The graph is reflected across the x-axis $\therefore a = -2$ (2 pt)

$\therefore y = \quad \therefore y = -2\cos(2x - \pi/2)$ (1 pt)

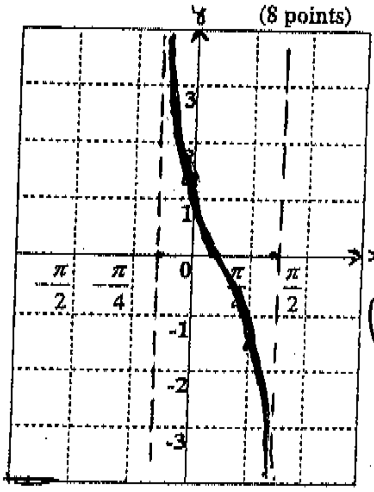
Q. 2: For the function $y = \frac{3}{2}\cot\left(2x + \frac{\pi}{4}\right)$

$a = 3/2, \ b = 2, \ c = +\pi/4$

a) Find the period and the phase shift of the graph. **(8 points)**

Period: $\frac{\pi}{b} = \frac{\pi}{2}$ (1 pt)

Phase shift: $\frac{-c}{b} = \frac{-\pi}{8}$ (1 pt)



($3\frac{1}{2}$ pts)

b) Sketch one full period of the graph.
(Show at least three points and two asymptotes on the graph of the function).

$x = -\frac{\pi}{8} + \frac{\pi}{4} = \frac{\pi}{8}$ is a zero

The graph passes through $(0, 3/2)$ and $(\frac{\pi}{4}, -3/2)$ $\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1\frac{1}{2} \text{ pts}\right)$

Asymptotes $x = -\frac{\pi}{8}$ and $x = \frac{3\pi}{8}$ $\left(\frac{1}{2} + \frac{1}{2} = 1 \text{ pt}\right)$

(students will show these on graph. On paper, calculations are not necessary)

Q. 3: Verify the identity $\dfrac{\cos x \tan x + 2\cos x - \tan x - 2}{\tan x + 2} = \cos x - 1$ **(7 points)**

$$\frac{\cos x \tan x + 2\cos x - \tan x - 2}{\tan x + 2}$$

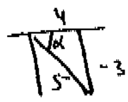
$$\frac{\cos x(\tan x + 2) - 1(\tan x + 2)}{\tan x + 2} \quad (3\text{ Pt})$$

$$\underbrace{\frac{(\tan x + 2)(\cos x - 1)}{\tan x + 2}}_{(3\text{ Pts})} = \cos x - 1 \quad (1\text{ Pt})$$

Q. 4: Find the exact value of $\sin\dfrac{\alpha}{2}$, **when** $\csc\alpha = \dfrac{-5}{3}$, $\dfrac{7\pi}{2} < \alpha < 4\pi$ **(7 points)**

$$\csc\alpha = -\frac{5}{3} \quad \therefore \sin\alpha = \frac{-3}{5} \quad (1\text{ Pt})$$

$$\therefore \cos\alpha = \frac{4}{5} \quad (2\text{ Pts})$$



$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} = \pm\sqrt{\frac{1-4/5}{2}} = \pm\frac{1}{\sqrt{10}} = \pm\frac{\sqrt{10}}{10} \quad (3\text{ Pts})$$

$$\sin\frac{\alpha}{2} = -\frac{\sqrt{10}}{10} \quad (1\text{ Pt}) \quad (\because \frac{7\pi}{4} < \frac{\alpha}{2} < 2\pi)$$

Q. 5: Rewrite the function $f(x) = \sin x - \sqrt{3}\cos x$ **in the form of** $k\sin(x+\alpha)$.
Then find the amplitude, the period and the phase shift of the graph of $f(x)$. **(8 points)**

$$f(x) = \sin x - \sqrt{3}\cos x$$

$$a = 1,\ b = -\sqrt{3},\quad k = \sqrt{a^2+b^2} = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2 \quad (1\text{ Pt})$$

$$\cos\alpha = 1/2 \quad \sin\alpha = -\sqrt{3}/2 \quad (1\text{ Pt})$$

$$\therefore \alpha = 5\pi/3 \text{ or } -\pi/3 \quad (1\text{ Pt})$$

$$f(x) = 2\sin(x + 5\pi/3) \quad (2\text{ Pt})$$

Amplitude: 2 (1 Pt), period: 2π → (1 Pt)

Phase Shift is $-\dfrac{5\pi}{3}$ or $\dfrac{5\pi}{3}$ to the left (1 Pt)

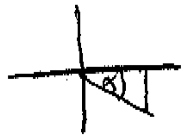
OR $f(x) = 2\sin(x - \pi/3)$

Phase Shift is $-\dfrac{c}{b} = \dfrac{\pi}{3}$

or $\dfrac{\pi}{3}$ to the right

Q. 6: Evaluate $\sin\left(2\sin^{-1}\left(\frac{-3}{5}\right)\right)$ **(8 points)**

$\alpha = \sin^{-1}\left(-\frac{3}{5}\right)$



α is in III quadrant

$\therefore \sin\alpha = -\frac{3}{5}$ (2Pts), $\cos\alpha = \frac{4}{5}$ (2 Pts)

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)$ — (2 Pts)

$= -\frac{24}{25}$ (2 Pts)

Q. 7: Solve the equation $\sin\frac{x}{2}+\cos x=1$ **for the exact solutions in the interval** $0\le x\le\pi$. **(8 points)**

$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

$\therefore \left(\pm\sqrt{\frac{1-\cos x}{2}}\right) + \cos x = 1$ (1 Pt)

$\pm\sqrt{\frac{1-\cos x}{2}} = 1-\cos x$

$\left(\sqrt{\frac{1-\cos x}{2}}\right)^2 = (1-\cos x)^2$

$\frac{1-\cos x}{2} = (1-\cos x)^2$

$(1-\cos x) = 2(1-\cos x)^2$

$(1-\cos x) - 2(1-\cos x)^2 = 0$

$(1-\cos x)\{1-2(1-\cos x)\} = 0$

$(1-\cos x)(-1+2\cos x) = 0$ (3 Pts)

Either $1-\cos x = 0 \Rightarrow \cos x = 1$ $x = 0$

or $-1+2\cos x = 0 \Rightarrow \cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$ — (2 Pts)

Both solutions satisfy the equation (1Pt)

$\therefore$ Solution in the interval $0\le x\le\pi$ are $0, \frac{\pi}{2}$ — (1Pt)

Q. 8: Given the vectors $U = \langle \sqrt{3}, 1 \rangle$ and $V = \langle 1, \sqrt{3} \rangle$, find **(10 points)**

a) The dot product of the vectors U and V.

$U \cdot V = \langle \sqrt{3}, 1 \rangle \cdot \langle 1, \sqrt{3} \rangle$ (2 Pts)

$= (\sqrt{3})(1) + (1)(\sqrt{3}) = 2\sqrt{3}$

b) The angle between the vectors U and V. (Total pts) (4)

$\cos\theta = \dfrac{U \cdot V}{\|U\|\,\|V\|}$. $\|U\| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ (1 Pt)

$\|V\| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$ (1 Pt)

$\therefore \cos\theta = \dfrac{2\sqrt{3}}{(2)(2)} = \underbrace{\sqrt{3}/2}_{(1\,Pt)}$ $\therefore \theta = \theta = \pi/6$ (1 Pt)

c) A vector W of magnitude 6 in the opposite direction of the vector $U - \sqrt{3}V$ (Total pts 4)

$U - \sqrt{3}\,V = \langle \sqrt{3}, 1 \rangle - \sqrt{3}(\langle 1, \sqrt{3} \rangle)$

$= \langle \sqrt{3}, 1 \rangle - \langle \sqrt{3}, 3 \rangle$

$= \langle \sqrt{3} - \sqrt{3}, 1 - 3 \rangle$

$= \langle 0, -2 \rangle$ (1 Pt)

$\|U - \sqrt{3}\,V\| = \sqrt{0^2 + (-2)^2} = 2$ (1 Pt)

A unit vector in the direction of $U - \sqrt{3}\,V$ is

$\dfrac{\langle 0, -2 \rangle}{2} = \langle 0, -1 \rangle$ (1 Pt)

$\therefore W = -6\langle 0, -1 \rangle = \langle 0, 6 \rangle$ (1 Pt)

Q. 9: Write the equation in standard form of the parabola that has vertex $(-4,1)$, axis of symmetry parallel to the y-axis and passing through the point $(-2,2)$. (8 points)

The equation of parabola with vertical axis of symmetry is

$(x-h)^2 = 4p(y-k)$ (1Pt), Vertex (h,k)

$(x+4)^2 = 4p(y-1)$ (2 Pts)

passes through $(-2, 2)$

$\therefore (-2+4)^2 = 4p(2-1)$ (2Pts) $\quad 4 = 4p \quad p = 1$ (1Pt)

$\therefore \quad (x+4)^2 = 4(y-1)$ (2 pts)

Q. 10: Consider the ellipse given by the equation **(8 points)**

$$9x^2 + y^2 + 18x - 6y + 9 = 0$$

a) Find its vertices and foci.

$9x^2 + 18x + y^2 - 6y + 9 = 0$

$9(x^2 + 2x + 1 - 1) + 1(y^2 - 6y + 9) = 0$

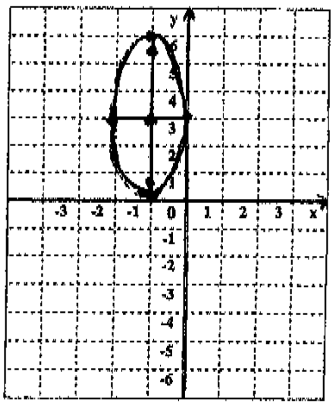
$9(x^2 + 2x + 1) - 9 + (y^2 - 6y + 9) = 0$

$9(x+1)^2 + (y-3)^2 = 9$

$\dfrac{(x+1)^2}{1} + \dfrac{(y-3)^2}{9} = 1$ (2Pts)

$a = 3,\ b = 1$, Centre (h,k)

~~**b) Sketch the graph of the ellipse.**~~



(2Pts)

Centre $(-1, 3)$

Vertices $(h, k \pm a)$,

$(-1, 3-3), (-1, 3+3)$

$(-1, 0), (-1, 6)$ (2Pts)

$c = \sqrt{a^2 - b^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$

foci : $(-1, 3-2\sqrt{2}), (-1, 3+2\sqrt{2}) \mapsto$ (2Pts)

Terminal points of minor axis $(-2, 3), (0, 3)$