King Fahd University of Petroleum and Minerals
College of Sciences
Prep-Year Math Program

KEY

Math 002 Exam II
Term 021 (2002-2003)
Sunday, December 22, 2002
Time Allowed: 90 Minutes

KEY

Student's Name: ______________________

ID #:__________________ **Section #:**____________________

This exam consists of Two parts

Part I : Multiple Choice: Bubble the correct answer on the OMR sheet.
Part II : Written Questions: Provide neat and complete solutions. Show all necessary steps for full credit.

Calculators, Pagers, or Mobiles are NOT allowed during this exam.

Question	Points	GRADER
Part I: MCQ (1 - 6)	12	
Part II: Written		
1	5	
2	3	
3	5	
4	3	
5	4	
6	4	
7	4	
8	4	

Total	
	44

Part I: (12-points) Multiple Choice Questions (MCQ).
Bubble the Correct Answer in the OMR Sheet.

1. The expression $2\csc x \cos\frac{x}{2}$ simplifies to

(a) $\csc\frac{x}{2}$

(b) $\sec\frac{x}{2}$

(c) $\tan\frac{x}{2}$

(d) $\cot\frac{x}{2}$

2. Which one of the following statements is **FALSE**?

(a) $\csc\left(\frac{\pi}{2}+\theta\right) = -\sec\theta$

(b) $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

(c) $\sec\left(\frac{\pi}{2}+\theta\right) = -\csc\theta$

(d) $\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$

3. The exact value of $\sin^{-1}\left(\sin\frac{9\pi}{5}\right)$ is

(a) $-\frac{\pi}{5}$

(b) $\frac{\pi}{5}$

(c) $\frac{9\pi}{5}$

(d) $-\frac{2\pi}{5}$

4. Which one of the following statements is always TRUE for any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ and any nonzero real number k?

(a) The vector $\dfrac{-\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector

(b) The vectors $\mathbf{u}$ and $k\mathbf{u}$ have the same direction

(c) $\|\mathbf{u}+\mathbf{v}\| < \|\mathbf{u}\| + \|\mathbf{v}\|$

(d) $\|k\mathbf{v}\| = k\|\mathbf{v}\|$

5. The value of $\dfrac{\tan\frac{7\pi}{12} - \tan\frac{3\pi}{4}}{1+\tan\frac{7\pi}{12}\tan\frac{3\pi}{4}}$ is equal to

(a) $-\tan\dfrac{\pi}{6}$

(b) $\tan\dfrac{\pi}{6}$

(c) $-\tan\dfrac{4\pi}{3}$

(d) $\tan\dfrac{4\pi}{3}$

6. The number of solutions of the equation $2-2|\cos x| = 1$ with $0 \le x \le \dfrac{3\pi}{2}$ is equal to

(a) 3

(b) 4

(c) 1

(d) 2

Part II: Written Questions.
[Provide neat and complete solution. Show all necessary steps for full credit.]

1. (5-points) Given the function $f(x) = -2\sin\left(2x - \frac{\pi}{4}\right)$.

   (a) Find the period of $f(x)$.

   The period $= \frac{2\pi}{2} = \pi$ --- 1 point

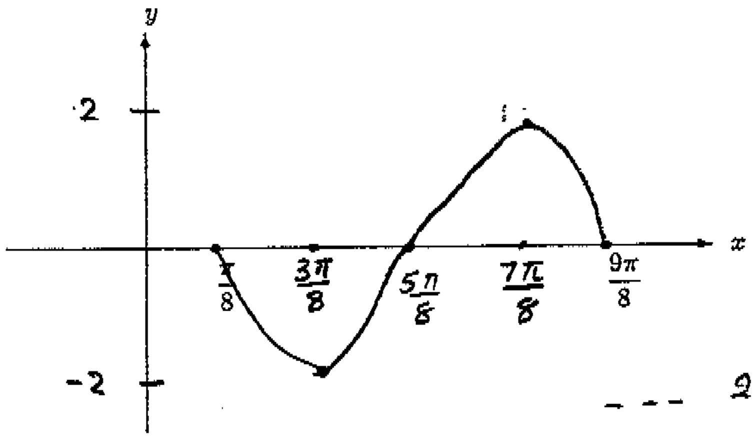
   (b) Find the phase shift of the graph of $f(x)$.

   The phase shift $= \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}$ --- 1 point

   (c) Find the range of $f(x)$.

   The range $= [-2, 2]$ --- 1 point

   (d) Sketch the graph of $f(x)$ over the interval $\left[\frac{\pi}{8}, \frac{9\pi}{8}\right]$.



--- 2 points

2. (3-points) If $0 \le \alpha < 2\pi$, find the exact value of $\sec\frac{\alpha}{2}$ given that $\cos\alpha = \frac{4}{5}$ and α is in quadrant IV.

$\frac{3\pi}{2} < \alpha < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{\alpha}{2} < \pi \Rightarrow \alpha \in Q\,II$ --- 1 point

$\Rightarrow \sec\frac{\alpha}{2} = \frac{1}{\cos\frac{\alpha}{2}} = -\sqrt{\frac{2}{1+\cos\alpha}}$ --- 1 point

$\Rightarrow \sec\frac{\alpha}{2} = -\sqrt{\frac{2}{1+\frac{4}{5}}} = -\sqrt{\frac{10}{9}} = -\frac{\sqrt{10}}{3}$ --- 1 point

3. (5-points) Given the vectors $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 5\mathbf{j}$.

(a) Find a vector of length 3 in the opposite direction of the vector $\mathbf{u}$.

The required vector $= \dfrac{-3u}{\|u\|}$ ... 1 point

$= \dfrac{6i - 9j}{\sqrt{4+9}}$

$= \dfrac{6}{\sqrt{13}}\, i - \dfrac{9}{\sqrt{13}}\, j$ --- 1 point

(b) Find the measure of the smallest angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Let α be the required angle $\Longrightarrow$

$\cos\alpha = \dfrac{u \cdot v}{\|u\|\ \|v\|}$ --- 1 point

$= \dfrac{(-2)(1) + (3)(5)}{\sqrt{4+9}\ \sqrt{1+25}} = \dfrac{13}{\sqrt{13}\ \sqrt{26}} = \dfrac{13}{13\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ --- 1 point

$\Longrightarrow \alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$ (or 45°) --- 1 point

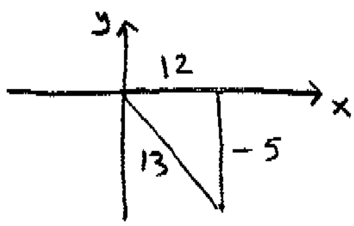
4. (3-points) Solve $\sin^{-1} x - \tan^{-1}\left(-\frac{5}{12}\right) = \pi/2$

$\Rightarrow \sin^{-1} x = \frac{\pi}{2} + \tan^{-1}\left(-\frac{5}{12}\right)$

$\Rightarrow \sin(\sin^{-1} x) = \sin\left(\frac{\pi}{2} + \tan^{-1}\left(-\frac{5}{12}\right)\right)$ ... 1 point

$\Rightarrow x = \cos\left(\tan^{-1}\left(-\frac{5}{12}\right)\right)$ --- 1 point

$= \dfrac{12}{13}$ ... 1 point

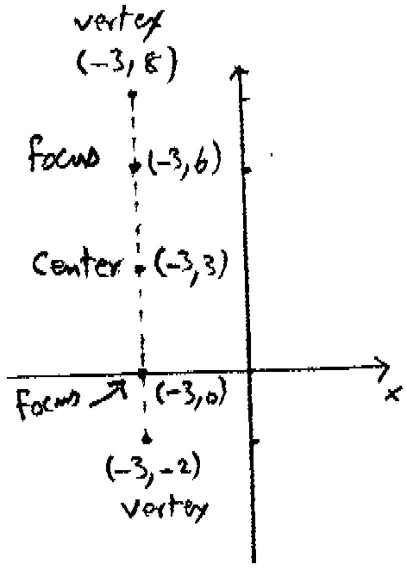


5. (4-points) Find the standard form of the equation of the ellipse that has foci at $(-3, 0)$ and $(-3, 6)$ and vertices at $(-3, -2)$ and $(-3, 8)$.

$\Longrightarrow$ The major axis is parallel to the y-axis and the center is at $\left(-3, \frac{0+6}{2}\right) \equiv (-3, 3)$ --- 1 point

$a = 8 - 3 = 5$
$c = 3 - 0 = 3$ --- 1 point

$b^2 = a^2 - c^2 = 25 - 9 = 16$ --- 1 point



$\Longrightarrow$ The required equation is :

$$\frac{(x+3)^2}{16} + \frac{(y-3)^2}{25} = 1$$

6. (4-points) Verify the identity $\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \csc x - \cot x, \quad 0 < x < \dfrac{\pi}{2}$.

$$\sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} = \sqrt{\frac{(1-\cos x)^2}{\sin^2 x}} \quad \text{--- 1 point}$$

$$= \frac{1-\cos x}{\sin x}, \ \left(\text{since } 0 < x < \frac{\pi}{2}\right) \quad \text{--- 1 point}$$

$$= \frac{1}{\sin x} - \frac{\cos x}{\sin x} \quad \text{--- 1 point}$$

$$= \csc x - \cot x \quad \text{--- 1 point}$$

as required.

7. (4-points) Find the vertex, focus, and directrix of the parabola given by the equation $6y - 3x^2 - 12x + 4 = 0$.

Put the equation in the standard form:

$$3x^2 + 12x = 6y + 4 \Rightarrow 3(x^2 + 4x) = 6y + 4 \Rightarrow$$

$$3(x+2)^2 = 6y + 4 + 12 \Rightarrow 3(x+4)^2 = 6y + 16$$

$$\Rightarrow 3(x+2)^2 = 6\left(y + \frac{8}{3}\right) \Rightarrow$$

$$\boxed{(x+2)^2 = 2\left(y + \frac{8}{3}\right) \quad \text{--- 1 point}}$$

Compare with $(x-h)^2 = 4p(y-k) \Rightarrow$

$\boxed{\text{The coordinates of the vertex are } \left(-2, -\frac{8}{3}\right) \text{ ... 1 point}}$

and $4p = 2 \Rightarrow p = \frac{1}{2} \Rightarrow$

$\boxed{\text{The coordinates of the focus are } \left(-2, -\frac{8}{3} + \frac{1}{2}\right) = \left(-2, \frac{-13}{6}\right) \text{ --- 1 point}}$

$\boxed{\text{The equation of the directrix is } y = -\frac{8}{3} - \frac{1}{2} = \frac{-19}{6} \text{ --- 1 point}}$

8. (4-points) Solve $2\sin x - \cos 2x = \frac{1}{2}$, where $0 \le x < \pi$.

$\Rightarrow 2\sin x - (1 - 2\sin^2 x) = \frac{1}{2}$ --- 1 point

$\Rightarrow 4\sin x - 2 + 4\sin^2 x = 1$

$\Rightarrow 4\sin^2 x + 4\sin x - 3 = 0$

$\Rightarrow (2\sin x - 1)(2\sin x + 3) = 0$ --- 1 point

$\Rightarrow \sin x = \frac{1}{2}$ or $\cancel{\sin x = -\frac{3}{2}}$ --- 1 point

$\Rightarrow$ The solutions in the interval $0 \le x < \pi$ are: $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. --- 1 point