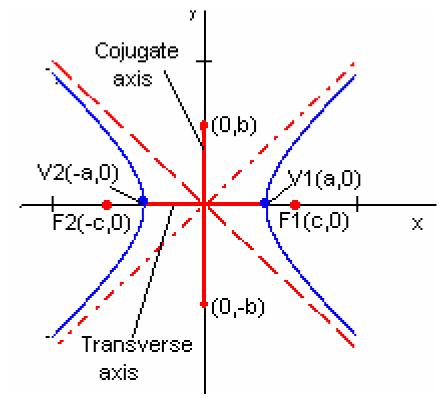
8.3 Hyperbolas

Definition of a Hyperbola

A **hyperbola** is the set of all points in the plane, the difference between whose distances from fixed two points



- The transverse axis is the line segment joining the vertices, and its length is denoted by 2a.
- The midpoint of the transverse is the center of the hyperbola.
- The conjugate axis passes through the center of hyperbola and perpendicular to the transverse axis, and its length denoted by 2b.
- The distance between the two foci is denoted by 2c.
- Each hyperbola has two asymptotes that pass through the center of the hyperbola.

Standard Form of the Equation of a Hyperbola with Center at (h, k)

1. Transverse Axis Parallel to the x-axis

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

The coordinates of the vertices are $V1(h+a,k)$ and $V2(h-a,k)$. The coordinates of foci are $F1(h+c,k)$ and $F2(h-c,k)$, where $c^2=a^2+b^2$.
The equations of the asymptotes are

$$y-k=\pm\frac{b}{a}(x-h).$$

2. Transverse Axis Parallel to the y-axis

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

The coordinates of the vertices are $V1(h,k+a)$ and $V2(h,k-a)$. The coordinates of foci are $F1(h,k+c)$ and $F2(h,k-c)$, where $c^2=a^2+b^2$.

The equations of the asymptotes are $y-k=\pm\frac{a}{b}(x-h)$.

Notes: To find the equation of asymptotes of a hyperbola in standard form replace 1 by 0 and solve for y.

Example # 1 Find the vertices, foci, center, and asymptotes of each hyperbola.

a) $\frac{(x+6)^2}{9}-\frac{9y^2}{4}=1$

Solution

Rewrite in standard form $\frac{(x+6)^2}{9}-\frac{y^2}{4/9}=1$

$$\textit{Horizontal center } (-6,0)$$

$$a=3,\ b=\frac{2}{3},\ c^2=a^2+b^2=9+\frac{4}{9}=\frac{85}{9}\to c=\frac{\sqrt{85}}{3}$$

$$\textit{Vertices } (-6+3,0)=(-3,0),(-6-3,0)=(-9,0)$$

$$\textit{Foci} \left(-6+\frac{\sqrt{85}}{3},0\right),\left(-6-\frac{\sqrt{85}}{3},0\right)$$

To find asymptotes replace 1 *by* 0 *and solve for y*

$$\frac{(x+6)^2}{9}-\frac{y^2}{4/9}=0$$

$$\frac{(x+6)^2}{9}=\frac{y^2}{4/9}$$

$$y^2=\frac{4}{9\cdot 9}(x+6)^2$$

$$y=\pm\frac{2}{9}(x+6).$$

b) $4x^2-25y^2+16x+50y-109=0$. Graph the hyperbola

Solution

$$4x^2+16x-25y^2+50y=109$$

$$4[x^2+4x]-25[y^2-2y]=109$$

$$4\left[x^2+4x+2^2-2^2\right]-25\left[y^2-2y+1^2-1^2\right]=109$$

$$4\left[(x+2)^2-4\right]-25\left[(y-1)^2-1\right]=109$$

$$4(x+2)^2-16-25(y-1)^2+25=109$$

$$4(x+2)^2-25(y-1)^2=100$$

$$\frac{(x+2)^2}{25}-\frac{(y-1)^2}{4}=1$$

$$a=5,\ b=2\quad c^2=a^2+b^2=25+4=29\to c=\sqrt{29}$$

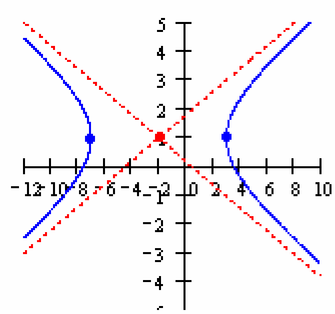
$$\textit{Horizontal hyperbola center } (-2,1)$$

$$\textit{Vertices } (-2+5,1)=(3,1),\ (-2-5,1)=(-7,1)$$

$$\textit{Foci } (-2+\sqrt{29},1),(-2-\sqrt{29},1)$$

$$\textit{Asymptotes } \frac{(x+2)^2}{25}-\frac{(y-1)^2}{4}=0$$

$$\frac{(y-1)^2}{4}=\frac{(x+2)^2}{25}\to y=\pm\frac{2}{5}(x+2)+1.$$



c) $16y^2 - 9x^2 + 32y - 36x - 164 = 0$. H.W.
Do exr.3, 15, 25, page 604.

Example # 2 Find the equation in standard form of each hyperbola.
a) Vertices (6,3) and (2,3), foci (7,3) and (1,3)
Solution

Since both vertices lie on the horizontal line,
the transverse is horizontal, the standard equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Length of transverse axis = distance between vertices

$$2a = |6-2| = 4$$

$$a = 2$$

The center (h,k) is the midpoint of the vertices

$$(\text{h,k}) = \left(\frac{6+2}{2}, \frac{3+3}{2}\right) = (4,3)$$

$c =$ distance between the center and focus $= |7-4| = 3$

$$c^2 = a^2 + b^2 \rightarrow 9 = 4 + b^2 \rightarrow b^2 = 5$$

Then the standard form is

$$\frac{(x-4)^2}{4} - \frac{(y-3)^2}{5} = 1.$$

b) Foci (-2, 1) and (-2, 7), slope of an asymptote 5/4.

Solution

Since foci lie on the vertical line, the transverse axis is vertical.

The standard equation is $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$

$$\text{Center}\left(\frac{-2+-2}{2},\frac{1+7}{2}\right)=(-2,4)$$

$$c=\text{distance between the center and focus}$$

$$=|7-4|=3$$

To find equation of an asymptotes

$$\frac{(y-4)^2}{a^2}-\frac{(x+2)^2}{b^2}=0\rightarrow(y-4)^2=\frac{a^2(x+2)^2}{b^2}$$

$$y-4=\pm\frac{a}{b}(x+2)$$

$$\text{slope }=\frac{5}{4}=\frac{a}{b}\rightarrow a=\frac{5b}{4}\rightarrow a^2=\frac{25b^2}{16}$$

$$c^2=a^2+b^2\rightarrow 9=\frac{25b^2}{16}+b^2$$

$$9=\frac{41b^2}{16}$$

$$b^2=\frac{144}{41}\rightarrow\ a^2=\frac{25\frac{144}{41}}{16}=\frac{225}{41}$$

Then the standard equation is

$$\frac{(y-4)^2}{\frac{225}{41}} - \frac{(x+2)^2}{\frac{144}{41}} = 1.$$

Eccentricity (e) of a Hyperbola

The eccentricity e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is the length of the semi-transverse axis.

$$e = \frac{c}{a}$$

Do exr. 33, 39, 44, 45, page 605.