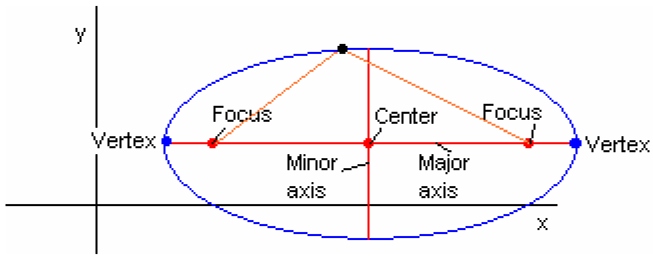
8.2 Ellipses

Definition of an Ellipse

An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points (foci) is a positive constant.



Notes:

- The graph of an ellipse has two axis of symmetry
  1. The longer axis is called the major axis, and its length denoted by 2a. The foci of the ellipse are on the major axis.
  2. The shorter axis is called minor axis, and its length denoted by 2b.
- The center of the ellipse is the midpoint of the major axis.
- The endpoints of the major axis are the vertices of the ellipse.

Standard Form of the Equation of an Ellipse with Center at (h,k)

1. Major Axis Parallel to x-axis

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\quad a>b$$

The length of major axis is 2a. The length of minor axis is 2b.

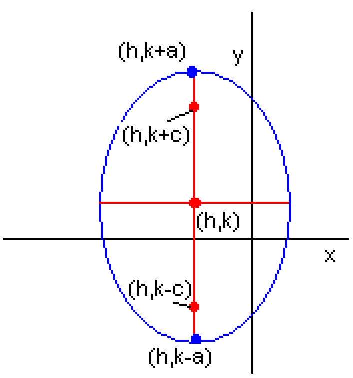
The coordinates of the vertices are $(h+a,k)$ and $(h-a,k)$, and the coordinates of the foci are $(h+c,k)$ and $(h-c,k)$, where $c^2=a^2-b^2$.

2. Major Axis Parallel to y-axis

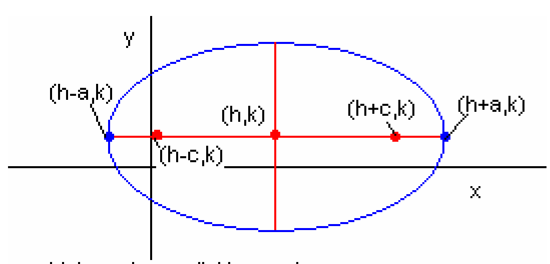
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\quad a>b$$

The length of major axis is 2a. The length of minor axis is 2b.

The coordinates of the vertices are $(h,k+a)$ and $(h,k-a)$, and the coordinates of the foci are $(h,k+c)$ and $(h,k-c)$, where $c^2=a^2-b^2$.



Major axis parallel to y-axis



Major axis parallel to x-axis

Example #1 Find the center, vertices, and foci of the ellipse. Sketch the graph.

a) $\dfrac{(x+2)^2}{25}+\dfrac{y^2}{16}=1$

Solution

$Horizontal\ ellipse\ center\,(-2,0)$

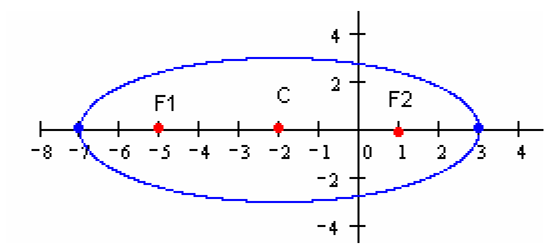
$a=5,\ b=4$

$c^2=a^2-b^2=25-16=9$

$c=3$

$The\ vertices\ are\,(-2+5,0)=(3,0)\ and\ (-2-5,0)=(-7,0)$

$The\ foci\ are\,(-2+3,0)=(1,0)\ and\ (-2-3,0)=(-5,0)$



b) $16x^2+9y^2-64x-54y+1=0$

Solution

First rewrite the equation in standard form by completing the square.

$16x^2-64x+9y^2-54y+1=0$

$16(x^2-4x)+9(y^2-6y)=-1$

$$16\left[x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2\right]+9\left[y^2-6y+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2\right]=-1$$

$16\left[(x-2)^2-4\right]+9\left[(y-3)^2-9\right]=-1$

$16(x-2)^2-64+9(y-3)^2-81=-1$

$16(x-2)^2+9(y-3)^2=-1+64+81$

$16(x-2)^2+9(y-3)^2=144$ $\quad$ $*\,divide\ by\ 144$

$$\frac{(x-2)^2}{9}+\frac{(y-3)^2}{16}=1$$

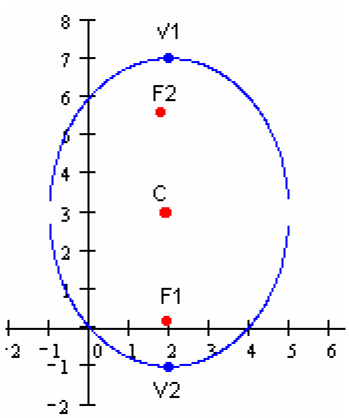
Vertical ellipse center $(2,3)$

$a=4,\ b=3$

$c^2=a^2-b^2=16-9=7\quad c=\sqrt{7}$

Vertices $(2,3+4)=(2,7)$ *and* $(2,3-4)=(2,-1)$

Foci $(2,3+\sqrt{7})$ *and* $(2,3-\sqrt{7})$



Do exr.7, 9, 29, and 31, page 602.

Example #2 Find the standard form of the equation of the ellipse with foci at (6,4) and (-2,4), and major axis of length 10.

Solution

Because foci are horizontal the major axis is horizontal.
The length of major axis is 2a. Thus 2a=10 and $a=5$.
The center is the midpoint between two foci. Therefore,(2,4).
The distance between the center and a focus is c. Therefore, $c=4$.
To find b^2*, use the equation* $c^2=a^2-b^2$.

$16=25-b^2$

$b^2=9$

Thus the equation in standard form is

$$\frac{(x-2)^2}{25}+\frac{(y-4)^2}{9}=1.$$

Do exr. 33, 36, 38, and 39, page 605.

Eccentricity (e) of an Ellipse

The eccentricity e of an ellipse is the ratio of c to a, where c is the distance from the center to the focus and a is one-half the length of the major axis. That is,

$$e=\frac{c}{a}$$

Example #3 Find the standard form of the equation of the ellipse with vertices at (3,4) and (3,0), and eccentricity $\frac{1}{4}$.

Solution

Because vertices are vertical the major axis is vertical.
The distance between the vertices is 2a. Therefore $a=2$.

$$e=\frac{c}{a}\rightarrow\frac{1}{4}=\frac{c}{2}\rightarrow c=\frac{1}{2}$$

$$c^2=a^2-b^2\rightarrow\frac{1}{4}=4-b^2\rightarrow b^2=4-\frac{1}{4}=\frac{15}{4}$$

Center = midpoint between two vertices $=(3,2)$.

Thus the equation in standard form is

$$\frac{(x-3)^2}{\frac{15}{4}}+\frac{(y-2)^2}{4}=1$$

Do exr. 47, 49, and 51, page 605.