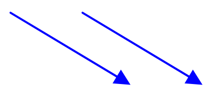
7.3 Vectors

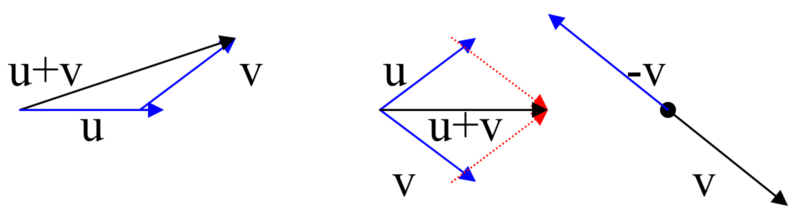
- Quantities that possess both magnitude and direction are called vector quantities.
- A vector is a directed line segment. The length of the line segment is the magnitude of the vector, and the direction of the vector is measured by an angle.
- The vector $\overrightarrow{OP}$ from point O to point P, O called initial point (or tail) of the vector, P called terminal point (or head) of the vector.

.

- The magnitude of the vector $\vec{v}$ is denoted by $\|v\|$.
- Equivalent vectors have the same magnitude and the same direction



- Vector addition:



u+v called resultant

- Components of vector:
  Let $p_1(x_1, y_1)$ be the initial point of a vector and $p_2(x_2, y_2)$ its terminal point. Then an equivalent vector $\vec{v}$ has initial point at the origin and terminal point $p(a,b)$, where $a = x_2 - x_1$ *and* $b = y_2 - y_1$. The vector $\vec{v}$ can denoted by $v = \langle a,b \rangle$; a and b are called the components of the vector $\vec{v}$.

Example #1 Find the components of a vector whose tail is the point $A(2,-1)$ and its head is $B(-2,4)$. Write an equivalent vector v in terms of its components.

Solution

$$a=-2-2=-4, \qquad and \qquad b=4-(-1)=5$$

$v=\langle -4,5\rangle$.

Do exr. 3 and 5 page 566.

Fundamental Vector Operations

If $v=\langle a,b\rangle$ and $w=\langle c,d\rangle$ are two vectors and k is a real nubmer, then

1. $\|\mathrm{v}\|=\sqrt{a^2+b^2}$ *magnitude of v.*
2. $v+w=\langle a,b\rangle+\langle c,d\rangle=\langle a+c,b+d\rangle$
3. $kv=\langle ka,kb\rangle$ *scalar multiplication.*

Example #2 Given $v=\langle 2,-3\rangle$ and $w=\langle 3,1\rangle$, find

a) $\|w\|$ b) $v+w$ b) $\|2v-3w\|$.

Solution

a) $\|w\|=\sqrt{3^2+1^2}=\sqrt{10}$

b) $v+w=\langle 2,-3\rangle+\langle 3,1\rangle=\langle 2+3,-3+1\rangle=\langle 5,-2\rangle$

c) First we find $2v-3w$

$$2v-3w=2\langle 2,-3\rangle-3\langle 3,1\rangle$$
$$=\langle 4,-6\rangle-\langle 9,3\rangle=\langle -5,-9\rangle$$

Then $\|2v-3w\|=\sqrt{(-5)^2+(-9)^2}=\sqrt{25+81}=\sqrt{105}$

- A unit vector is a vector whose magnitude is 1.

  For example, the vector $v=\left\langle -\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle$ is a unit vector

  because $\|v\|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$.

- Given any vector v, we can obtain a unit vector in the direction of v by dividing each component of v by the magnitude of v, $\|v\|$.

Example #3 Find a unit vector u in the direction of $v = \langle 2, -3 \rangle$

Solution

$$\|v\| = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

$$u = \left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle = \left\langle \frac{2\sqrt{13}}{13}, \frac{-3\sqrt{13}}{13} \right\rangle$$

Definition of Unit Vectors i and j

$$i = \langle 1, 0 \rangle \qquad j = \langle 0, 1 \rangle$$

- If v is a vector and $v = \langle a_1, a_2 \rangle$, then $v = a_1 i + a_2 j$

Example #4 Given $v = 2i - 3j$ and $w = \langle -2, 8 \rangle$, find $-2v + \frac{1}{2}w$.

Solution

$$w = -2i + 8j$$

$$-2v + \frac{1}{2}w = -2(2i - 3j) + \frac{1}{2}(-2i + 8j)$$
$$= -4i + 6j - i + 4j$$
$$= -5i + 10j$$

Direction Angle for Vector v

Let $v = \langle a_1, a_2 \rangle = a_1 i + a_2 j$, $v \neq 0$. Then

$$a_1 = \|v\| \cos\theta \qquad \text{and} \qquad a_2 = \|v\| \sin\theta$$

where θ is the angle between the positive x-axis and v.

a_1 is the horizontal component, and a_2 is the vertical component

$\frac{a_2}{a_1} = \frac{\|v\|\sin\theta}{\|v\|\cos\theta} = \tan\theta \to \theta = \tan^{-1}\frac{a_2}{a_1}$, is direction angle of v.

Example #5 Find horizontal and vertical component of a vector v that has magnitude 9 and direction angle $\frac{2\pi}{3}$.

Solution

$$a_1 = 9\cos\frac{2\pi}{3} = 9(-\frac{1}{2}) = -\frac{9}{2}$$

$$a_2 = 9\sin(\frac{2\pi}{3}) = 9\frac{\sqrt{3}}{2}$$

$$v = -\frac{9}{2}i + \frac{9\sqrt{3}}{2}j$$

Do exr. 35 page 566

xample #6 Find the magnitude and direction angle of the vector

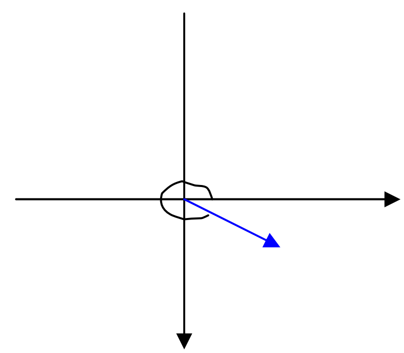
a) $v = \sqrt{3}i - j.$ b) $u = -i - \sqrt{3}j$

Solution

$$\|v\| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

$$direction\ angle\ \theta = \tan^{-1}(\frac{-1}{\sqrt{3}})$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$



b) H.W.

Definition of Dot Product

Given $v = \langle a,b \rangle$ and $w = \langle c,d \rangle$, the dot product of v and w is givenby

$$v \cdot w = ac + bd$$

Example #7 Find the dot product of $v = 3i + 5j$ and $w = \langle -2,4 \rangle$.

Solution

$$v \cdot w = 3(-2) + 5 \cdot 4 = -6 + 20 = 14$$

Properties of Dot Product

In the following properties, u v, and w are vectors and a is a scalar.

1. $v \cdot w = w \cdot v$
2. $u \cdot (v + w) = u \cdot v + u \cdot w$
3. $a(u \cdot v) = (au) \cdot u \cdot (av)$
4. $v \cdot v = \|v\|^2$
5. $0 \cdot v = 0$
6. $v = \langle a, b \rangle$, then $\|v\| = \sqrt{v \cdot v}$

Alternative formula for the Dot Product

If v and w are two nonzero vectors and α is the smallest non-negative angle between v and w, then $v \cdot w = \|v\|\|w\|\cos\alpha$, and

$$\alpha = \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right).$$

Example #7 Find the angle between two vectors $v = 2i + 2j$, and $w = -i + j$.

Solution

$$\cos\alpha = \frac{v \cdot w}{\|v\|\|w\|}$$

$$= \frac{2(-1) + 2 \cdot 1}{\sqrt{2^2 + 2^2}\sqrt{(-1)^2 + 1^2}} = \frac{0}{\sqrt{8}\sqrt{2}} = 0$$

$$\alpha = \cos^{-1}(0) = \frac{\pi}{2}.$$

Definition of The Scalar Projection of v on w

If v and w are two nonzero vectors and α is the smallest positive angle between v and w, then the scalar projection of v on w, $proj_w v$, is given by

$$proj_w v = \|v\|\cos\alpha$$

- To derive an alternate formula for $proj_w v$, consider the dot product, $v \cdot w = \|v\|\|w\|\cos\alpha$. Solving for $\|v\|\cos\alpha$, which is $proj_w v$, we have $$proj_w v = \frac{v \cdot w}{\|w\|}$$

Example #8 Given $v = -2i - 3j$, and $w = i + 4j$, find $proj_w v$.

Solution

$$proj_w v = \frac{v \cdot w}{\|w\|} = \frac{-2 \cdot 1 + (-3) \cdot 4}{\sqrt{1^2 + 4^2}} = \frac{-14}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}} = -\frac{14\sqrt{17}}{17}$$

- Two nonzero vectors are parallel when the angle α between the vectors is 0° or 180°.
- Two vectors are perpendicular (orthogonal) when the angle α between the vectors is 90°.

Example #8 Find the value of k if the following vectors are perpendicular.

$v = \langle 3k, -2 \rangle, \quad u = 4i + 3j.$

Solution

u and v are perpendicular $\rightarrow \mathrm{u} \cdot \mathrm{v} = 0$

$$3k(4) + (-2)(3) = 0 \rightarrow k = \frac{6}{12} = \frac{1}{2}.$$