6.5 Inverse Trigonometric Function

- In this section we define an inverse function of trigonometric function.
- For example $f(x)=\sin x$, because the graph of $y=\sin x$ is not one-to-one function. So f does not have an inverse function.
- $f(x)=\sin x$ over the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, the graph of f is one-to-one function. So f has an inverse function.
- $y=\sin^{-1}x$ is read "y is the inverse sine of x."
- Some textbooks use the notation $\arcsin x$ instead of $\sin^{-1}x$.

Definition of $\sin^{-1}x$

$$y=\sin^{-1}x \quad \text{if and only if} \quad x=\sin y$$

where $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le \frac{\pi}{2}$.

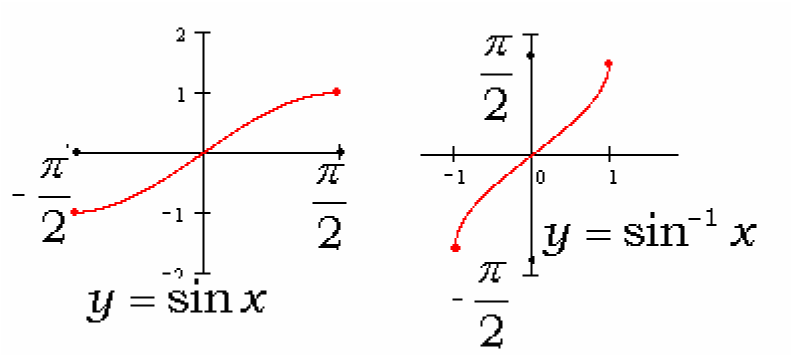


Table 7.2 Inverse Trigonometric Functions

Function	Domain	Range	Quadrant
$y=\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	I, IV
$y=\csc^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	I, IV
$y=\cos^{-1}x$	$[-1,1]$	$[0,\pi]$	I, II
$y=\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$	I, II
$y=\tan^{-1}x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	I, IV
$y=\cot^{-1}x$	$(-\infty,\infty)$	$(0,\pi)$	I, II

Example #1 Find the exact value of the following

a) $y=\sin^{-1}(\frac{1}{2})$ b) $y=\tan^{-1}(-1)$ c) $y=\cos^{-1}(-\frac{1}{2})$

Solution

a) $y=\sin^{-1}(\frac{1}{2})$, is the angle in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$. Therefore $y=\frac{\pi}{6}$.

b) $y=\tan^{-1}(-1)$, is the angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is -1. Therefore $y=-\frac{\pi}{4}$.

c) $y=\cos^{-1}(-\frac{1}{2})$, is the angle in the interval $[0,\pi]$ whose cosine is $-\frac{1}{2}$. Therefore $y=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Do exr. 9, 15, and 19, page 513.

Identities for $\csc^{-1} x, \sec^{-1} x, and \tan^{-1} x$

If $x \le -1$ or $x \ge 1$, then

$$\csc^{-1}x = \sin^{-1}\frac{1}{x} \quad \text{and} \quad \sec^{-1}x = \cos^{-1}\frac{1}{x}$$

If x is a real number, then

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\frac{1}{x}, & for\ x > 0 \\ \tan^{-1}\frac{1}{x} + \pi, & for\ x < 0 \\ \frac{\pi}{2}, & for\ x = 0 \end{cases}$$

Composition of Trigonometric Function and Their Inverses

- $\sin(\sin^{-1} x) = x\ and\ \cos(\cos^{-1} x) = x \quad if\ -1 \le x \le 1$
- $\tan(\tan^{-1} x) = x,\ for\ any\ real\ number\ x.$
- $\sin^{-1}(\sin x) = x,\ if\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$
- $\cos^{-1}(\cos x) = x,\ if\ 0 \le x \le \pi.$
- $\tan^{-1}(\tan x) = x,\ if\ -\frac{\pi}{2} < x < \frac{\pi}{2}.$

Example #2 Find the exact value of the following

a) $\sin^{-1}(\sin\frac{\pi}{5})$ b) $\tan(\tan^{-1} - 4)$ c) $\cos^{-1}(\cos\frac{7\pi}{5})$

d) $\sin(\sin^{-1}\frac{\sqrt{5}}{2})$ e) $\tan^{-1}(\tan\frac{7\pi}{5})$

Solution

a) $\sin^{-1}(\sin\frac{\pi}{5}) = \frac{\pi}{5}$, becuase $\frac{\pi}{5}$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b) $\tan(\tan^{-1} - 4) = -4$, becuase -4 is a real number.

c) $\cos^{-1}(\cos\frac{7\pi}{5})$,

$\frac{7\pi}{5}$ is not in the interval $[0,\pi]$; however, the reference angle

for $\frac{7\pi}{5}$ is $\frac{2\pi}{5}$, becuase the value of $\cos\frac{7\pi}{5}$ is negative,

the angle is in Quadrant II. So $\theta = \pi - \frac{2\pi}{5} = \frac{3\pi}{5}$ which is

in the interval $[0,\pi]$, $\cos^{-1}(\cos\frac{7\pi}{5}) = \cos^{-1}(\cos\frac{3\pi}{5}) = \frac{3\pi}{5}$.

d)

$\sin(\sin^{-1}\frac{\sqrt{5}}{2})$, becuase $\frac{\sqrt{5}}{2}$ is not in the domain $\sin^{-1}x$,

$\sin(\sin^{-1}\frac{\sqrt{5}}{2})$ is undefined.

e) $\tan^{-1}(\tan\frac{7\pi}{5})$

$\frac{7\pi}{5}$ is not in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; however, the reference angle

for $\frac{7\pi}{5}$ is $\frac{2\pi}{5}$, becuase the value of $\tan\frac{7\pi}{5}$ is positive,

the angle is in Quadrant I. So $\theta = \frac{2\pi}{5}$ which is in the interval

$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\tan^{-1}(\tan\frac{7\pi}{5}) = \tan^{-1}(\tan\frac{2\pi}{5}) = \frac{2\pi}{5}$.

Do exr.21, 31, and 32 , page 513.

Example #3 Find the exact value of the following

a) $\sin(2\sin^{-1}\frac{3}{4})$ b) $\cos\left(\sin^{-1}(-\frac{1}{3})+\cos^{-1}\frac{2}{5}\right)$.

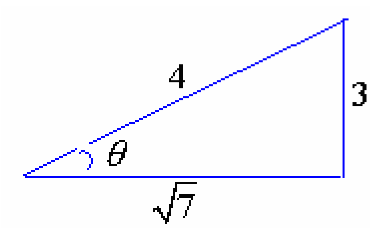
Solution

a) $\sin(2\sin^{-1}\frac{3}{4})$

$$Let\ \theta=\sin^{-1}\frac{3}{4}\ \rightarrow\sin(2\sin^{-1}\frac{3}{4})=\sin(2\theta)=2\sin\theta\cos\theta$$

$$\theta=\sin^{-1}\frac{3}{4}, \text{take sine both sides}\rightarrow\sin\theta=\sin(\sin^{-1}\frac{3}{4})$$

$$\sin\theta=\frac{3}{4}$$



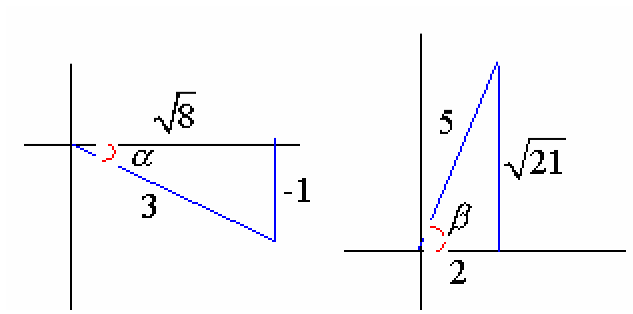
$$\sin 2\theta=2\frac{3}{4}\frac{\sqrt{7}}{4}=\frac{3\sqrt{7}}{8}$$

b)

$$Let\ \alpha=\sin^{-1}(-\frac{1}{3}), and\ \ \beta=\cos^{-1}\frac{2}{5}.Thus$$

$$\sin\alpha=-\frac{1}{3}\ and\ \ \cos\beta=\frac{2}{5}$$

$$\cos\left(\sin^{-1}(-\frac{1}{3})+\cos^{-1}\frac{2}{5}\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$



$$\cos\left(\sin^{-1}(-\frac{1}{3})+\cos^{-1}\frac{2}{5}\right)=\frac{2}{5}\frac{\sqrt{8}}{3}-\frac{-1}{3}\frac{\sqrt{21}}{5}=\frac{2\sqrt{8}+\sqrt{21}}{15}$$

Do exr.26, 51, 53, and 56, page 513.

Example #4 Solve the equation $\cos^{-1}\frac{1}{3}+\sin^{-1}x=\frac{\pi}{2}$ for x.

Solution H.W.

Example #5 Verify the identity $\tan^{-1}x+\tan^{-1}\frac{1}{x}=\frac{\pi}{2},\ x>0$

Solution

$$Let\ \alpha=\tan^{-1}x\ \ and\ \ \beta=\tan^{-1}\frac{1}{x}$$

$$\to\tan\alpha=x\ \ and\ \ \tan\beta=\frac{1}{x}$$

$$\tan^{-1}x+\tan^{-1}\frac{1}{x}=\alpha+\beta$$

$$= \tan^{-1}\left[\tan\left(\alpha+\beta\right)\right]$$

$$= \tan^{-1}\left[\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right]$$

$$= \tan^{-1}\left[\frac{x+\frac{1}{x}}{1-x\frac{1}{x}}\right] = \tan^{-1}\left[\frac{\frac{x^2+1}{x}}{1-1}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x^{2+1}}{x}}{0}\right], \text{which is undefined}$$

$$= \frac{\pi}{2}$$

Do exr.72 and 74, page 514.

H.W. Evaluate the expression $y = \tan(\cos^{-1} x)$

Example #6 Graph $y = \tan^{-1}(x+1) - 2$

First graph $y = \tan^{-1} x$, then shifting one unit to the left and two units downward.