6.3 Double and Half-Angle Identities

Double-Angle Identities

$$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$$
$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$$

Using $\cos^2\alpha+\sin^2\alpha=1,$ we can rewrite $\cos 2\alpha$ as follows:

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha \qquad *\cos^2\alpha=1-\sin^2\alpha$$
$$* \qquad \cos 2\alpha=(1-\sin^2\alpha)-\sin^2\alpha$$
$$\cos 2\alpha=1-2\sin^2\alpha.$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha \qquad *\sin^2\alpha=1-\cos^2\alpha$$
$$* \qquad =\cos^2\alpha-(1-\cos^2\alpha)$$
$$\cos 2\alpha=2\cos^2\alpha-1$$

$$* \qquad \tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$
$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

Example #1 Write each expression in terms of a single trigonometric function.

a) $\cos^2 6\alpha-\sin^2 6\alpha$

Solution

$$\cos^2 6\alpha-\sin^2 6\alpha=\cos 2(6\alpha) \qquad *\textit{Use } \cos 2\beta=\cos^2\beta-\sin^2\beta$$
$$=\cos 12\alpha \qquad \textit{with } \beta=6\alpha$$

b) $\sin 3\theta \cos 3\theta$

Solution

$$\sin 3\theta \cos 3\theta = \frac{1}{2}(2\sin 3\theta \cos 3\theta) \qquad *Use \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$= \frac{1}{2}(\sin 2(3\theta)) = \frac{1}{2}\sin 6\theta \qquad with\, \alpha = 3\theta$$

Do exr. 5, 4, and 7, page 491.

Example #2 Find the exact value of $\sin 2\theta$ and $\tan 2\theta$,

if $\tan\theta = \frac{4}{3}$, θ in Quadrant III.

Solution

$$\tan\theta = \frac{4}{3} \rightarrow y = -4, x = -3, r = \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{-4}{5},\ \cos\theta = \frac{x}{r} = \frac{-3}{5}.$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{-4}{5}\frac{-3}{5} = \frac{12}{25}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}}$$

$$= \frac{\frac{8}{3}}{\frac{-7}{9}} = \frac{8}{3}\frac{9}{-7} = -\frac{24}{7}$$

Do exr. 34 and 36, page 492.

Half-Angle Identities

- From double-angle identity

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad *Let\ \alpha/2 = \theta$$

$$\cos\alpha = 1 - \sin^2\alpha/2$$

Solve for $\sin\alpha/2$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

The sign $\pm$ depending on which quadrant $\alpha/2$ lies.

- In similar manner, we derive $\cos\alpha/2$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

- Two different identities for $\tan\alpha/2$. See page 489

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

Example #3 Find the exact value of $\sin\theta/2$, $\cos\theta/2$ and $\tan\theta/2$

if $\csc\theta = -\frac{5}{3}$, θ is in Quadrant IV

Solution

Because θ is in Quadrant IV $\rightarrow 3\pi/2 < \theta < 2\pi$

$\rightarrow 3\pi/4 < \theta/2 < \pi$, then $\theta/2$ is in Quadrant II

$$\csc\theta = -\frac{5}{3} \rightarrow \sin\theta = -\frac{3}{5}$$

$$\cos^2\theta + \left(-\frac{3}{5}\right)^2 = 1 \rightarrow \cos\theta = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+4/5}{2}}$$

$$= -\sqrt{\frac{1+\frac{4}{5}}{2}} = -\sqrt{\frac{9}{10}} = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-4/5}{2}}$$

$$= \sqrt{\frac{5-4}{10}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{4}{5}}{\frac{-3}{5}} = \frac{5-4}{-3} = -\frac{1}{3}$$

Example #4 Verify the given identity.

a) $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

Solution
$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{1+2\cos^2 x-1}$$
$$= \frac{2\sin x\cos x}{2\cos^2 x}$$
$$= \frac{\sin x}{\cos x} = \tan x$$

b) $2\cos^4 x - \cos^2 x - 2\sin^2 x\cos^2 x + \sin^2 x = \cos^2 2x$

Solution
$$2\cos^4 x - \cos^2 x - 2\sin^2 x\cos^2 x + \sin^2 x$$
$$= \cos^2 x(2\cos^2 x - 1) - \sin^2 x(2\cos^2 x - 1)$$
$$= (2\cos^2 x - 1)(\cos^2 x - \sin^2 x)$$
$$= \cos 2x \cdot \cos 2x = \cos^2 2x$$

c) $\cos^2\frac{x}{2} - \cos x = \sin^2\frac{x}{2}$

Solution

$$\cos^2\frac{x}{2} - \cos x$$

$$= \left(\pm\sqrt{\frac{1+\cos}{2}}\right)^2 - \cos x$$

$$= \frac{1+\cos x}{2} - \cos x$$

$$= \frac{1+\cos x - 2\cos x}{2}$$

$$= \frac{1-\cos x}{2}$$

$$= \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 = \sin^2\frac{x}{2}$$

d) $\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 = 1 + \sin x$

Solution $\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$

$$= \cos^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} + \sin^2\frac{x}{2}$$

$$= \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \sin 2(\frac{x}{2})$$

$$= 1 + \sin x$$

Do exr. 53, 63, 71, 82, and 99, pages 442 and 493.