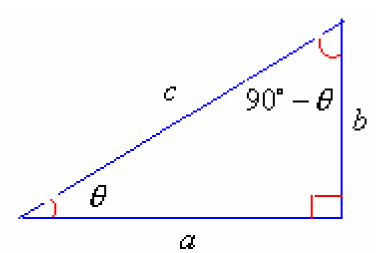
6.2 Sum, Difference, and Cofunction Identities

Sum and Difference Identities

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$



$$\sin\theta=\frac{b}{c}=\cos(90^\circ-\theta)$$
$$\tan\theta=\frac{b}{a}=\cot(90^\circ-\theta)$$
$$\sec\theta=\frac{c}{a}=\csc(90^\circ-\theta)$$

Cofunction Identities

$\sin\theta=\cos(90^\circ-\theta)$ $\quad\cos\theta=\sin(90^\circ-\theta)$

$\tan\theta=\cot(90^\circ-\theta)$ $\quad\cot\theta=\tan(90^\circ-\theta)$

$\sec\theta=\csc(90^\circ-\theta)$ $\quad\csc\theta=\sec(90^\circ-\theta)$

If θ is in radian measure, relpace 90° with $\pi/2$.

To verify $\sin(90^\circ - \theta) = \cos\theta$, we use the identity for $\sin(\alpha - \beta)$.

$$\begin{aligned}\sin(90^\circ - \theta) &= \sin 90^\circ \cos\theta - \cos 90^\circ \sin\theta \\ &= 1 \cdot \cos\theta - 0 \cdot \sin\theta \\ &= \cos\theta.\end{aligned}$$

To verify $\cos(90^\circ - \theta) = \sin\theta$, we use the identity for $\cos(\alpha - \beta)$.

$$\begin{aligned}\cos(90^\circ - \theta) &= \cos 90^\circ \cos\theta + \sin 90^\circ \sin\theta \\ &= 0 \cdot \cos\theta + 1 \cdot \sin\theta \\ &= \sin\theta.\end{aligned}$$

To verify $\tan(90^\circ - \theta) = \cot\theta$, we use the ratio identity.

$$\tan(90^\circ - \theta) = \frac{\sin(90^\circ - \theta)}{\cos(90^\circ - \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

To verify $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$, see page 481.

Example #1 Find the exact value of the expression.

a) $\cos(\frac{\pi}{4} - \frac{\pi}{3})$ b) $\sin 212^\circ \cos 122^\circ - \cos 212^\circ \sin 122^\circ$.

Solution

a)

$$\begin{aligned}\cos(\frac{\pi}{4} - \frac{\pi}{3}) &= \cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ &= \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} \\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b)

$$\sin 212^\circ \cos 122^\circ - \cos 212^\circ \sin 122^\circ$$
$$= \sin(212^\circ - 122^\circ)$$
$$= \sin 90^\circ$$
$$= 1.$$

Do exr. 11, 14, 16, 17 page 484.

Example #2 Given $\sin\alpha = -\frac{4}{5}$, α in Quadrant III, and $\cos\beta = -\frac{12}{13}$, β in Quadrant II, find

a) $\tan(\alpha+\beta)$ b) $\sec(\alpha-\beta)$

Solution

a)

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta},$$

$\sin\alpha = -\frac{4}{5}$, to calculate $\cos\alpha$

$$\cos^2\alpha + \left(-\frac{4}{5}\right)^2 = 1$$

$$\cos^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$$

$\cos\alpha = -\frac{3}{5}$, α in Quadrant III,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{3}$$

$$\cos\beta = -\frac{12}{13}, \text{to calculate } \sin\beta$$

$$\sin^2\beta + \left(-\frac{12}{13}\right)^2 = 1$$

$$\sin^2\beta = 1 - \frac{144}{169} = \frac{25}{169}$$

$$\sin\beta = \frac{5}{13}, \beta \text{ in Quadrant II,}$$

$$\tan\beta = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{4}{3} + (-\frac{5}{12})}{1 - \frac{4}{3}(-\frac{5}{12})} = \frac{\frac{11}{12}}{1 + \frac{5}{9}} = \frac{\frac{11}{12}}{\frac{14}{9}} = \frac{11}{12}\frac{9}{14} = \frac{33}{56}$$

b) $\sec(\alpha - \beta) = \dfrac{1}{\cos(\alpha - \beta)}$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= -\frac{3}{5}\left(\frac{-12}{13}\right) + \left(-\frac{4}{5}\right)\frac{5}{13}$$

$$= \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$$

$$\sec(\alpha-\beta) = \frac{65}{16}$$

Do exr. 33 and 34 page 484.

Example #3 Verify the identity.

a) $\cos\left(\frac{\pi}{2}+\alpha+\beta\right)=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

Solution

$$\cos\left(\frac{\pi}{2}+\alpha+\beta\right)=\cos\left[\frac{\pi}{2}+(\alpha+\beta)\right]$$
$$=\cos\frac{\pi}{2}\cos(\alpha+\beta)-\sin\frac{\pi}{2}\sin(\alpha+\beta)$$
$$=(0)\cos(\alpha+\beta)-(1)\sin(\alpha+\beta)$$
$$=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Do exr. 63 and 65, page 485.

b) $\frac{\sin(x+h)-\sin x}{h}=\cos x\frac{\sin(\text{h})}{h}+\sin x\frac{\cos(\text{h})-1}{h}$

Solution

$$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x\cos\text{h}+\cos x\sin\text{h}-\sin x}{h}$$
$$=\frac{\sin x\cos\text{h}-\sin x+\cos x\sin\text{h}}{h}$$
$$=\frac{\sin x(\cos\text{h}-1)+\cos x\sin\text{h}}{h}$$
$$=\frac{\sin x(\cos\text{h}-1)}{h}+\frac{\cos x\sin\text{h}}{h}$$
$$=\sin x\frac{\cos\text{h}-1}{h}+\cos x\frac{\sin\text{h}}{h}$$

Do exr. 81 and 83, page 485.