10.4 Determinants

Definition of a Determinant of order 2

For a given 2×2 matrix

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of A is $\det(A)=|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc.$

Example #1 Evaluate the determinant of each the following matrices.

a) $A=\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ b) $B=\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

Definition of Minors and Cofactors

The **minor** M_{ij} of the element a_{ij} of a square matrix A of order $n\geq 3$ is the determinant of the matrix of order $n-1$ obtained by deleting the i^{th} row and the j^{th} column of A.

The **cofactor** C_{ij} of the element a_{ij} of a square matrix A is given by $C_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}.

Note: $C_{ij}=\left\{\begin{array}{ll} M_{ij}, & i+j \text{ is an even integer} \\ -M_{ij}, & i+j \text{ is an odd integer} \end{array}\right\}$

Example #2 Given

$$A=\begin{bmatrix} 6 & -3 & 2 \\ 0 & 1 & -2 \\ 2 & 4 & 5 \end{bmatrix}, \text{find } M_{21}, C_{21}, \text{ and } C_{22}$$

Cofactor Expansion

Given the square matrix A of order $n \geq 3$, the value of the determinant of A is the sum of the product of the elements of any row or column and their cofactor. For i^{th} row of A,

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

For j^{th} column of A,

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example # 3 Evaluate the determinant of

$$A=\begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 5 \\ -3 & -4 & -2 \end{bmatrix} \text{ by expanding by cofactor.}$$

Zero Expansion Property

If every element in one row (or one column) of a square matrix A is zero, then $|A| = 0$.

Interchange Property

If the matrix B is obtained from a matrix A by interchanging any two rows (column) of A, then $|B| = -|A|$.

Multiplication Property

If the matrix B is obtained from a matrix A by multiplying each elements of a row (column) of A by the same number c, then $|B| = c|A|$.

Addition Property

If the matrix B is obtained from a matrix A by adding a multiple of one row (column) to another row (column), then $|B| = |A|$.

Determinant of a Matrix in Triangular Form

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$ be a square matrix of order n in triangular form, then $|A| = a_{11}a_{22}\ldots a_{nn}$.

Example #4 Evaluate $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & 1 & 5 \\ -2 & -4 & 1 & 6 \end{vmatrix}$

Note: $|I_n| = 1$

Product Property of Determinant

If A and B are square matrices of order n, then

$$|AB| = |A||B|$$

Existence of the Inverse of a Square Matrix

If A is a square matrix of order n, then A has a multiplicative inverse if and only if $|A| \neq 0$. Furthermore

$$|A^{-1}| = \frac{1}{|A|}$$

Example #5 If A and B are square matrices of order 3, and $|A| = 3,\ |B| = -2$, the evaluate

a) $|AB|$ b) $|B^{-1}|$ c) $|3A|$

Solution

a) $|AB| = |A||B| = 3(-2) = -6$

b) $|B^{-1}| = \frac{1}{|B|} = \frac{1}{-2} = -\frac{1}{2}$

c) $|3A| = 2^3(3) = 24$.