CHAPTER 10
MATRICES

10.2 The Algebra of Matrices

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is m by n matrix, $(A_{m\times n})$ with m rows and n columns is said to be of order $m\times n$ or dimension.
- a_{ij} denotes the element in i^{th} row and j^{th} column of a matrix A.

  For example $A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & -2 & 4 \end{bmatrix}$, has order 2×3.

  $a_{12} = 1,\ a_{22} = -2,\ a_{23} = 4$.
- The elements $a_{11}, a_{22}, a_{33}, \cdots$ are called the main diagonal of a matrix A.
- A matrix that has n rows and n columns is called square matrix.
- A matrix that has only one column is called column matrix.
- A matrix that has only one row is called row matrix.
- A square matrix that has zeros for all its elements off the main diagonal is called diagonal matrix

- Two matrices are equal if they have the same order and elements placed in corresponding position are equal. For example $\begin{bmatrix}1\\2\\3\end{bmatrix} \neq \begin{bmatrix}1 & 2 & 3\end{bmatrix}$.

Definition of Addition of Matrices

If A and B are matrices of order $m \times n$, then the sum of the matrices is the $m \times n$ matrix given by

$$A + b = \left[a_{ij} + b_{ij}\right].$$

Here is an example. Let A $= \begin{bmatrix}2 & -4\\6 & 9\\-3 & 5\end{bmatrix}$ and B $= \begin{bmatrix}3 & 6\\8 & 7\\-2 & 11\end{bmatrix}$. Then

A+B $= \begin{bmatrix}2+3 & -4+6\\6+8 & 9+7\\-3-2 & 5+11\end{bmatrix} = \begin{bmatrix}5 & 2\\14 & 16\\-5 & 16\end{bmatrix}$.

Now let C $= \begin{bmatrix}2 & -1\\5 & 4\end{bmatrix}$. Here A+C is not defined because order of A and C not the same.

- Given the matrix A $= \left[a_{ij}\right]$, the additive inverse of A is $-A = \left[-a_{ij}\right]$
- The $m \times n$ zero matrix, denoted by **O,** is the matrix whose elements are all zeros.

Properties of Matrix Addition

Given matrices A, B, C and the zero matrix O, each of order $m \times n$, then the following properties hold.

Commutative	$A+B=B+A$
Associative	$A+(B+C)=(A+B)+C$
Additive inverse	$A+(-A)=O$
Additive identity	$A+O=O+A=A$

- Given $m \times n$ matrix $A=\left[a_{ij}\right]$ and the real number c, then the scalar multiplication is $cA=\left[ca_{ij}\right]$.

Properties of Scalar Multiplication

Given real numbers a, b, and c and matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ each of order $m \times n$, then

$$(a+c)A=aA+cA$$
$$c(A+B)=cA+cB$$
$$a(bA)=(ab)A$$

Example #1 Given $A=\begin{bmatrix}-3 & 2 & 4\\ 0 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}5 & -8 & 6\\ 3 & 2 & 0\end{bmatrix}$.

Find $2A+3B$

Solution

$$2A+3B=2\begin{bmatrix}-3 & 2 & 4\\ 0 & -2 & 1\end{bmatrix}+3\begin{bmatrix}5 & -8 & 6\\ 3 & 2 & 0\end{bmatrix}$$
$$=\begin{bmatrix}-6 & 4 & 8\\ 0 & -4 & 2\end{bmatrix}+\begin{bmatrix}15 & -24 & 18\\ 9 & 6 & 0\end{bmatrix}=\begin{bmatrix}9 & -20 & 26\\ 9 & 2 & 2\end{bmatrix}$$

Definition of the Product of Two Matrices

Let $A=\left[a_{ij}\right]$ be a matrix of order $m\times n$, and $B=\left[b_{ij}\right]$ be a matrix of order $n\times p$. Then the product AB is the matrix of order $m\times p$ given by $AB=C=\left[c_{ij}\right]$, where each element c_{ij} is

$$c_{ij}=\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix}\begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

Example #2 Given

$$A=\begin{bmatrix} 2 & -2 \\ 3 & 1 \\ 4 & 6 \end{bmatrix}, B=\begin{bmatrix} -1 & 3 & 0 \\ 5 & 7 & 2 \\ 4 & -3 & 1 \end{bmatrix}, \text{and } C=\begin{bmatrix} 2 & 9 \\ 8 & 3 \end{bmatrix}. \text{ Find}$$

a) AC b) AB c) BA

Properties of Matrix Multiplication

Associative Given matrices A of order $m\times n$ and B of order $n\times p$, and C of order $p\times q$, then

$$A(BC)=(AB)C$$

Distributive Given matrices A and B of order $m\times n$, C of order $n\times p$, and D of order $p\times m$ then

$$(A+B)C=AC+BC$$
$$D(A+B)=DA+DB$$

- The identity matrix of order n, denoted by I_n is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$$

For example $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- $AI_n = I_n A = A$, where A of order n.
- $A^2 = AA$, where A is square matrix
- If A and B are matrices of order n, then discuss
  - $(A+B)^2 = A^2 + 2AB + B^2$.
  - $If \ AB = O, then \ A = O \ or \ B = O.$