

8.8 Improper Integrals

- Improper integrals with infinite intervals of integration:

$$\int_a^{\infty} f(x) dx, \int_{-\infty}^a f(x) dx, \int_{-\infty}^{\infty} f(x) dx$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-1}^2 \frac{dx}{x}, \int_0^3 \frac{dx}{x-1}, \int_0^{\pi} \tan x dx$$

- Both

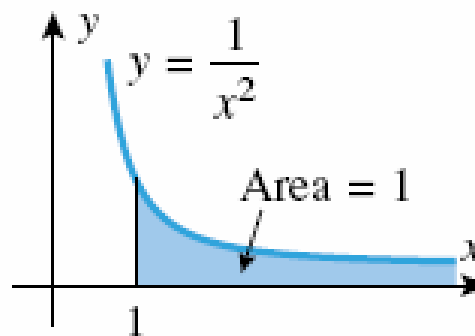
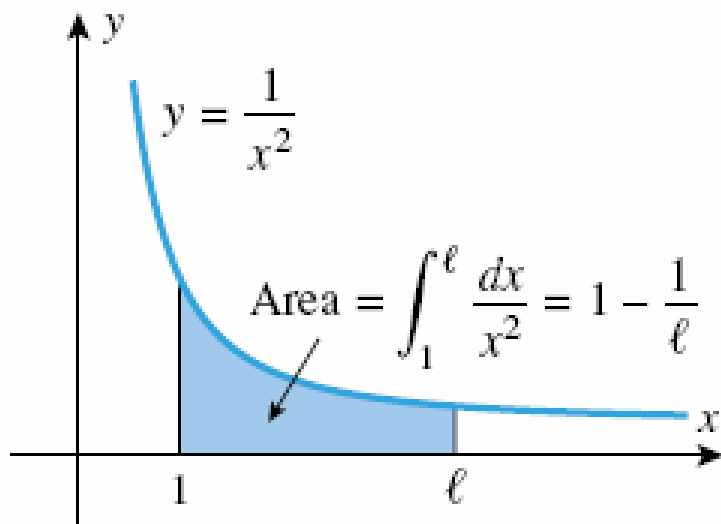
$$\int_{-1}^{\infty} \frac{dx}{x}, \int_{-\infty}^{\infty} \frac{dx}{x-1}, \int_0^{\infty} \sec x dx$$

8.8.1 DEFINITION. The *improper integral of f over the interval $[a, +\infty)$* is defined as

$$\int_a^{+\infty} f(x) dx = \lim_{\ell \rightarrow +\infty} \int_a^{\ell} f(x) dx$$

In the case where the limit exists, the improper integral is said to *converge*, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to *diverge*, and it is not assigned a value.

Example 1 Evaluate $\int_1^{\infty} \frac{dx}{x^2}$



Example 2 Evaluate $\int_1^{\infty} \frac{dx}{x}$

Example 3 For what value of p does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge?

Example 4 Evaluate $\int_2^{\infty} \frac{dx}{x \sqrt{\ln x}}$

8.8.3 DEFINITION. The *improper integral of f over the interval $(-\infty, b]$* is defined as

$$\int_{-\infty}^b f(x) dx = \lim_{k \rightarrow -\infty} \int_k^b f(x) dx \quad (2)$$

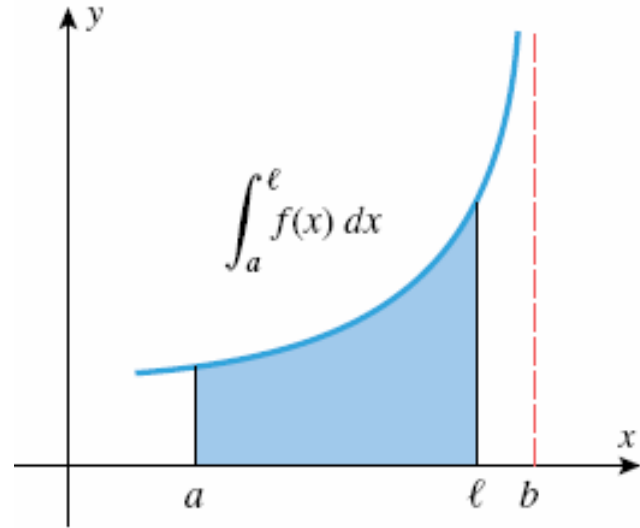
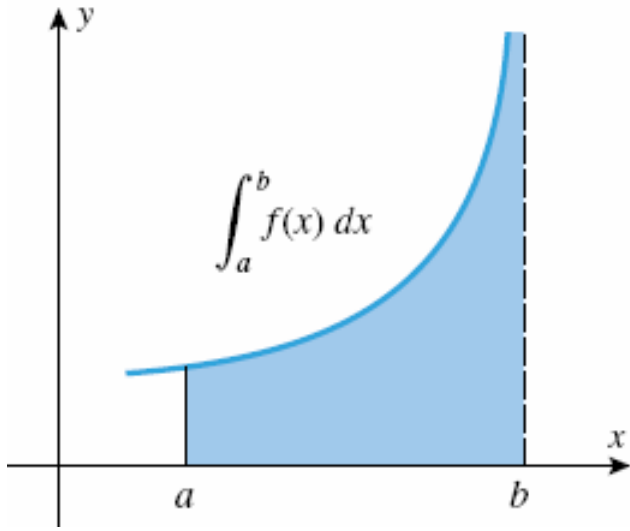
The integral is said to *converge* if the limit exists and *diverge* if it does not. The *improper integral of f over the interval $(-\infty, +\infty)$* is defined as

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx \quad (3)$$

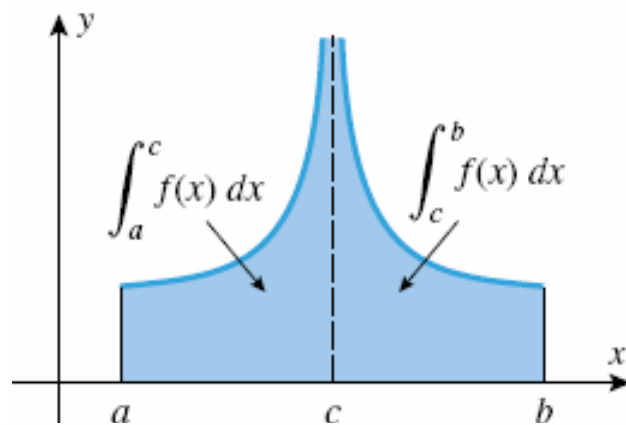
where c is any real number. The improper integral is said to *converge* if both terms converge and *diverge* if either term diverges.

Example 5 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Infinite Discontinuities



$$\int_a^b f(x) dx = \lim_{l \rightarrow b^-} \int_a^l f(x) dx$$



8.8.5 DEFINITION. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a , then the *improper integral of f over the interval $[a, b]$* is defined as

$$\int_a^b f(x) dx = \lim_{k \rightarrow a^+} \int_k^b f(x) dx$$

The integral is said to *converge* if the limit exists and *diverge* if it does not. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b) , then the *improper integral of f over the interval $[a, b]$* is defined as

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

The improper integral is said to *converge* if both terms converge and *diverge* if either term diverges (Figure 8.8.8).

Example 6 Evaluate

$$a) \int_0^3 \frac{dx}{\sqrt{3-x}}$$

$$b) \int_0^4 \frac{dx}{(x-3)^2}$$

$$c) \int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$$