8.8 Improper Integrals

- Improper integrals with infinite intervals of integration:

$$\int_{a}^{\infty} f(x)dx,\ \int_{-\infty}^{a} f(x)dx,\ \int_{-\infty}^{\infty} f(x)dx$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-1}^{2} \frac{dx}{x},\quad \int_{0}^{3} \frac{dx}{x-1},\quad \int_{0}^{\pi} \tan x\,dx$$

- Both

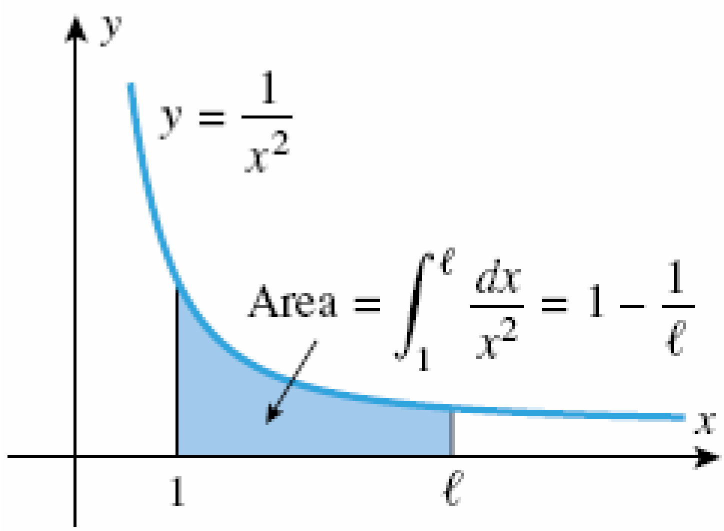
$$\int_{-1}^{\infty} \frac{dx}{x},\quad \int_{-\infty}^{\infty} \frac{dx}{x-1},\quad \int_{0}^{\infty} \sec x\,dx$$

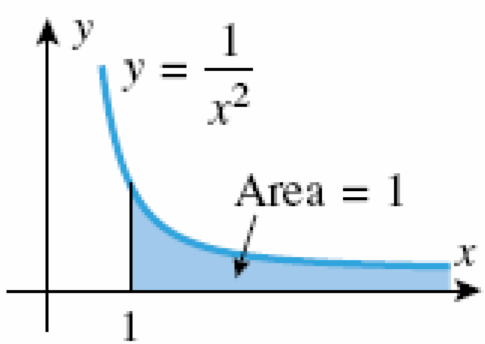
8.8.1 DEFINITION. The ***improper integral of f over the interval*** $[a, +\infty)$ is defined as

$$\int_a^{+\infty} f(x)\,dx = \lim_{\ell \to +\infty} \int_a^{\ell} f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 1 Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$





Example 2 Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x}$

Example 3 For what value of *p* does the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$ converge?

Example 4 Evaluate $\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}}$

8.8.3 DEFINITION. The ***improper integral of f over the interval*** $(-\infty, b]$ is defined as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{k\to-\infty} \int_{k}^{b} f(x)\,dx \tag{2}$$

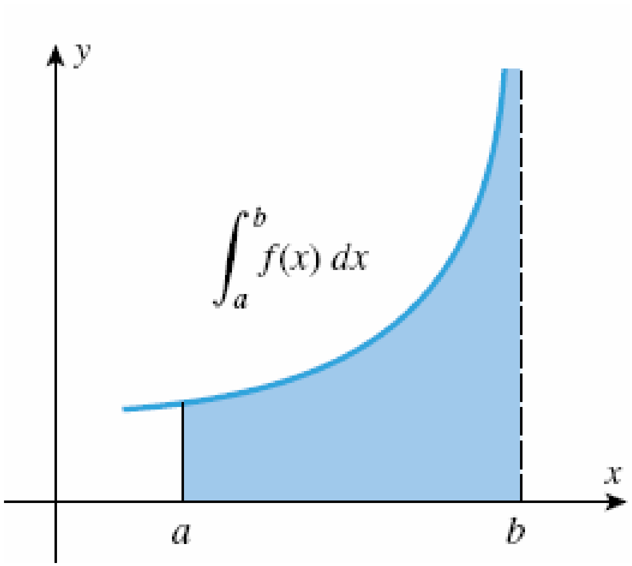
The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. The ***improper integral of f over the interval*** $(-\infty, +\infty)$ is defined as

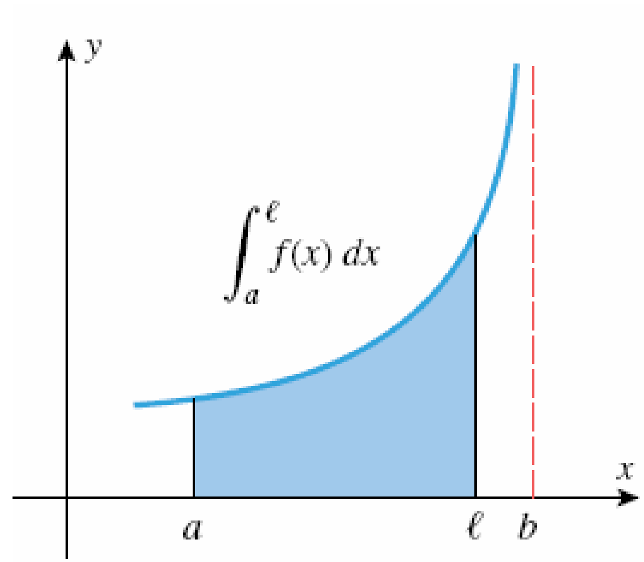
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if both terms converge and ***diverge*** if either term diverges.

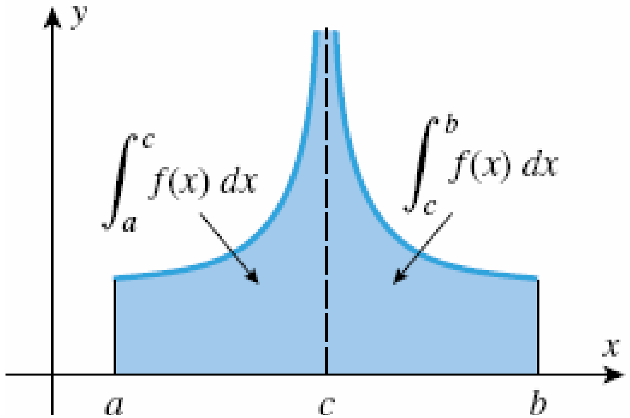
Example 5 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Infinite Discontinuities





$$\int_a^b f(x)dx = \lim_{l \to b^-} \int_a^l f(x)dx$$



8.8.5 DEFINITION. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval*** $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval*** $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The improper integral is said to ***converge*** if both terms converge and ***diverge*** if either term diverges (Figure 8.8.8).

Example 6 Evaluate

$$a)\int_0^3 \frac{dx}{\sqrt{3-x}} \qquad b)\int_0^4 \frac{dx}{(x-3)^2} \qquad c)\int_0^\infty \frac{dx}{\sqrt{x}\,(x+1)}$$