7.4 Length of a Plane Curve

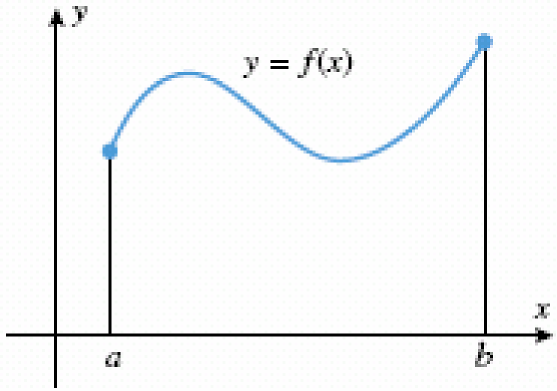
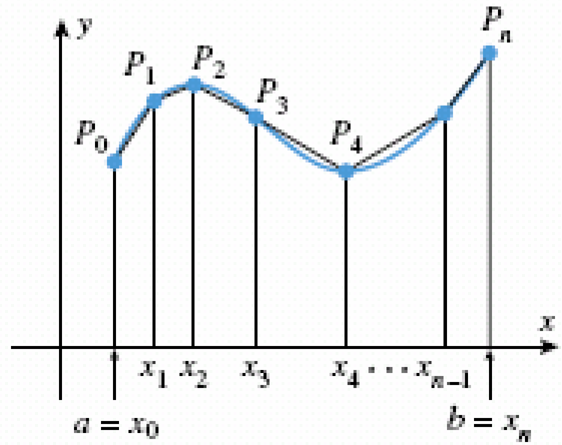
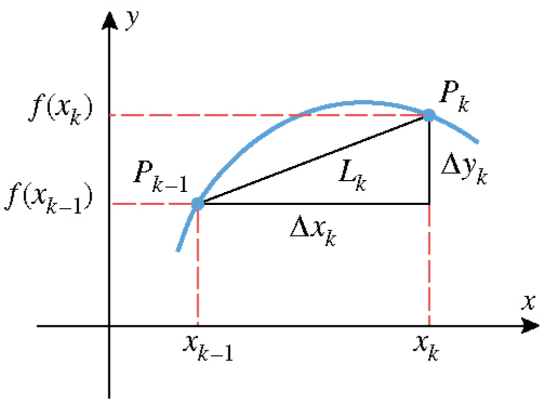


Figure 7.4.1





$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$L_k = \sqrt{(\Delta x_k)^2 + [f(\Delta x_k) - f(\Delta x_{k-1})]^2}$$

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(\Delta x_k) - f(\Delta x_{k-1})]^2}$$

$$MVT \rightarrow f'(x_k^*) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} \quad \Rightarrow L \approx \sum_{k=1}^{n} \sqrt{1 + \left[f'(x_k^*)\right]^2} \Delta x_k$$

$$\Rightarrow L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + \left[f'(x_k^*)\right]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

7.4.2 DEFINITION. If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{3}$$

$$OR \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Moreover, for the curve expressed in the form $x = g(y)$, where g' is continuous on [*a*, *b*], the arc length *L* from *y=c to y=d is*

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$$

Example 1

Find the arc length of the curve $f(x) = 3x^{2/3} - 10$

from the point *A*(8, 2) to *B*(27, 17)

Example 2

Find the arc length of the curve $24xy = y^4 + 48$ from $y = 2$ to $y = 4$

Parametric Curves:

For example graph

$a)\ \ x = 2t - 3,\ \ y = 6t - 7$

$b)\ x = \sin t,\ \ y = \sin^2 t,\ t \in \left[-\pi/2, \pi/2\right]$

Theorem. If a smooth curve has parametric equation

$$x = x(t),\ \ y(t) \quad a \le t \le b$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example 3 Exr.11/469