6.9 Logarithmic Function from the integral point of view

6.9.1 DEFINITION. The ***natural logarithm*** of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \tag{2}$$

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}dt\right] = \frac{1}{x} \qquad x > 0$$

6.9.2 THEOREM. *For any positive numbers a and c and any rational number r:*

(a) $\ln ac = \ln a + \ln c$ (b) $\ln \frac{1}{c} = -\ln c$

(c) $\ln \frac{a}{c} = \ln a - \ln c$ (d) $\ln a^r = r \ln a$

6.9.6 THEOREM.

(a) $\lim_{x\to 0}(1+x)^{1/x}=e$ (b) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$ (c) $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x=e$

Proof.

$$\frac{d}{dx}[\ln x]_{x=1}=\frac{1}{x}\Big|_{x=1}=1=\lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h}$$

$$1=\lim_{h\to 0}\frac{\ln(1+h)}{h}\to\lim_{x\to 0}\frac{1}{x}\ln(1+x)=1$$

$$\lim_{x\to 0}(1+x)^{1/x}=e^{\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right)}=e^{\lim_{x\to 0}\left(\ln(1+x)^{1/x}\right)}$$

$$=e^{\lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)}=e^1=e$$

Functions defined by integrals

Elementary Function; they include polynomial, rational, exponential, power, logarithmic, and trigonometric functions, and all other that can be obtained from these by addition, subtraction, multiplication, division, root and composition.

Other are not elementary.

The initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

has a solution of the form $y(x) = y_0 + \int_{x_0}^{x} f(t)dt$

Example1

$$\frac{dy}{dx} = xe^{x^2}, \quad y(0) = 0$$

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)dt\right] = \frac{d}{dx}\left[F(x)]_{h(x)}^{g(x)}\right] = \frac{d}{dx}\left[F(g(x)) - F(h(x))\right]$$

$$= F'(g(x))g'(x) - F'(h(x))h'(x) = f(g(x))g'(x) - f(h(x))h'(x)$$

Example 2

$$\frac{d}{dx}\int_{x^2}^{\sqrt{x}} \sin^2 t dt$$

Exercise 11a, 17, 34