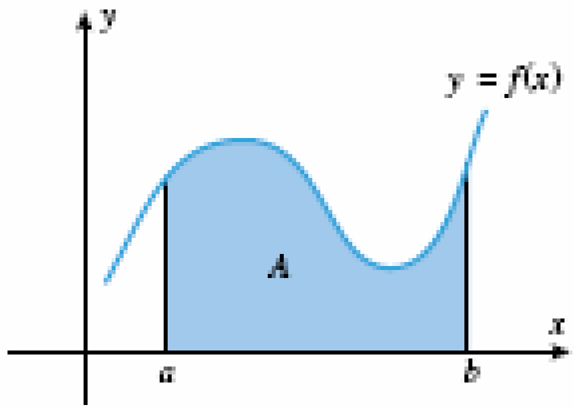
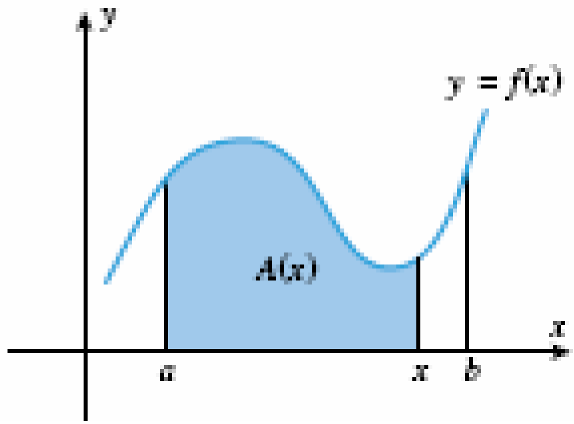
6.6 The Fundamental Theorem of Calculus



$$A = \int_a^b f(x)\,dx$$



$$A'(x) = f(x) \qquad A(a) = 0, \;\; A(b) = A$$

$A(x)$ *is antiderivative of* $f(x)$,

Then $F(x) = A(x) + c$

$$F(b) - F(a) = [A(b) + c] - [A(a) + c] = A(b) - A(a) = A - 0 = A$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

6.6.1 THEOREM *(The Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

We can express (2) as $\int_a^b f(x)dx = F(x)\Big]_a^b$

Example1 Evaluate

$a)\ \int_0^{\ln 2} 5e^x\,dx$ $\qquad b)\ \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}dx$ $\qquad \int_2^0 3x^2 dx$

Example2 Evaluate $\int_{-3}^{2} f(x)dx$

$$f(x) = \begin{cases} \sqrt{x}+1 & x \geq 0 \\ \dfrac{1}{x+4} & x < 0 \end{cases}$$

Total Area

If f is a continuous function on the interval [*a, b*], then we define the **total area** between the curve *y=f(x)* and the interval [*a, b*] to be

$$total\ area = \int_a^b |f(x)|dx$$

Example3 Find the total area between the curve $y = 1 - x^2$ And the *x-axis* over the interval [0, 2].

6.6.2 THEOREM (*The Mean-Value Theorem for Integrals*). *If f is continuous on a closed interval $[a, b]$, then there is at least one number x^* in $[a, b]$ such that*

$$\int_a^b f(x)\,dx = f(x^*)(b-a) \tag{7}$$

Note: $f(x^*)$ *called Average value*

Example 4

Find x^* in $[1,3]$ for $f(x)=x^2-1$ which satisfies the MVT for integrals.

6.6.3 THEOREM (*The Fundamental Theorem of Calculus, Part 2*). *If f is continuous on an interval I, then f has an antiderivative on I. In particular, if a is any number in I, then the function F defined by*

$$F(x)=\int_a^x f(t)\,dt$$

is an antiderivative of f on I; that is, $F'(x)=f(x)$ for each x in I, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x) \qquad (10)$$

Example 5

$$\frac{d}{dx}\left[\int_2^x t^2\,dt\right]=x^2 \qquad \frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right]=\frac{\sin x}{x}$$

HW $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = ?????$$

Note:

$$a)\ \int_{-a}^{a} f(x)dx = 0 \quad if \quad f(x)\ is\ odd$$

$$b)\ \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \quad if \quad f(x)\ is\ even$$