10.3 Infinite Series

$$\frac{2}{3} = 0.6666... = 0.6 + 0.06 + 0.006 + 0.0006 + ...$$

DEFINITION. An *infinite series* is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the *terms* of the series.

$$\frac{2}{3} = 0.6666... = 0.6 + 0.06 + 0.006 + 0.0006 + ...$$

$$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$$

$$S_1 = \frac{6}{10} = 0.6$$

$$S_2 = \frac{6}{10} + \frac{6}{10^2} = 0.66$$

$$S_3 = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} = 0.666$$

$$S_4 = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} = 0.6666$$

$$S_n = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \cdots + \frac{6}{10^n} \quad \{s_n\}_{n=1}^{\infty} \textit{ is called sequence of partial sum}$$

$$\frac{1}{10} S_n = \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \cdots + \frac{6}{10^{n+1}}$$

$$S_n - \frac{1}{10} S_n = \frac{6}{10} - \frac{6}{10^{n+1}} \longrightarrow \frac{9}{10} S_n = \frac{6}{10} - \frac{6}{10^{n+1}}$$

$$S_n = \frac{2}{3} - \frac{2}{3 \cdot 10^n} \longrightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{2}{3} - \frac{2}{3\cdot 10^n}\right) = \frac{2}{3}$$

In general

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$S_1 = u_1,\ S_2 = u_1 + u_2,\ S_3 = u_1 + u_2 + u_3$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k, \quad N^{th} \textit{ partial sum}$$

DEFINITION. Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to ***converge*** to S, and S is called the ***sum*** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to ***diverge***. A divergent series has no sum.

Example 1

Determine whether the following series converge. If it converge, find the sum

$a) \sum_{k=1}^{\infty} (-1)^k$ $\qquad$ $b) \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Geometric Series

THEOREM. *A geometric series*

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$$

converges if $|r| < 1$ *and diverges if* $|r| \geq 1$. *If the series converges, then the sum is*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Proof. $S_n = a + ar + ar^2 + \cdots + ar^n, \quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$

$$S_n - rS_n = a - ar^{n+1} \rightarrow S_n = \frac{a - ar^{n+1}}{(1-r)} \rightarrow \quad S_n = \frac{a}{(1-r)}\left(1 - r^{n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{(1-r)}\left(1 - r^{n+1}\right) = \frac{a}{1-r}, \ if \ |r| < 1$$

$$If \ |r| > 1 \rightarrow \lim_{n\to\infty} r^{n+1} = \pm\infty \rightarrow diverge$$

$$If \ r = 1 \rightarrow S_n = a + a + \cdots = a(1+n), which \ \ divrge$$

$$If \ r = -1 \rightarrow S_n = a - a + a \cdots, which \ \ divrge$$

Example 2

Determine whether the following series converge. If it converge, find the sum.

$$a)\ \frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots \qquad b)\ 1+x+x^2+x^3+\cdots \qquad c)\ \sum_{k=1}^{\infty} 2^{2k}7^{1-k}$$

$$d)\ \sum_{k=1}^{\infty} \frac{4^{k+1}}{3^k}$$

Example 3

Express the repeating decimal as a fraction. $0.123123123...$

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \quad \textit{Diverge}$$

Telescoping Series

$$\sum_{2}^{\infty} \frac{1}{k^2 - 1}$$