Chapter 10

Infinite Series

10.1 Sequences

A **sequence** is an unending succession of numbers, called **terms.** $a_1, a_2, a_3, ...$

Some examples 1,2,3,… $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$ 1,3,5,7,…

General Term of a Sequence

$$1, 3, 5, 7, ..., (2n-1), ...$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ..., \frac{1}{2^n}, ...$$

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, ..., (-1)^{n+1}\frac{n}{n+1}, ...$$

$$a_1, a_2, a_3, ..., a_n, ...$$

Brace Notation

General Term	*Brace Notation*
$1,3,5,7,\ldots,(2n-1),\ldots$	$\{2n-1\}_{n=1}^{\infty}$
$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots,\frac{1}{2^n},\ldots$	$\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$
$\frac{1}{2},-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\ldots,(-1)^{n+1}\frac{n}{n+1},\ldots$	$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{\infty}$
$a_1,a_2,a_3,\ldots,a_n,\ldots$	$\{a_n\}_{n=1}^{\infty}$

Example 1

$2,4,6,\ldots$ $\longrightarrow$ $\{2n\}_{n=1}^{\infty}$ or $\{2n+2\}_{n=0}^{\infty}$

10.2.1 DEFINITION. A ***sequence*** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots$.

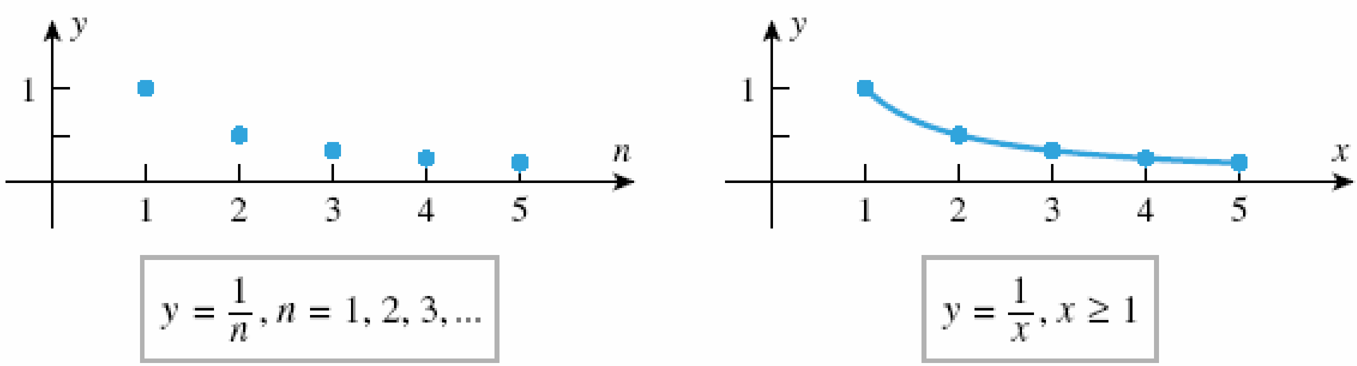
Some examples

$$1, 3, 5, \ldots, (2n-1) \to f(n) = 2n-1 \qquad f(1) = 1,\ f(4) = 7, \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots \to f(n) = \frac{1}{n} \qquad f(1) = 1,\ f(5) = \frac{1}{5}, \ldots$$

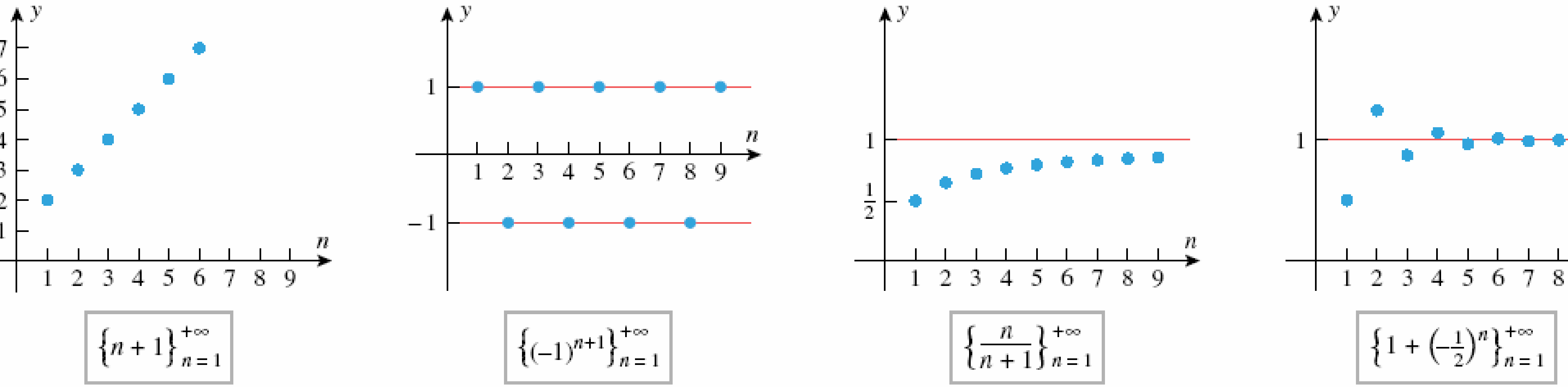
Graphs of Sequences

$$y = \frac{1}{n}, \quad n = 1, 2, 3, \ldots \longrightarrow y = \frac{1}{x}, \quad x \geq 1$$



(a) (b)

Limit of a Sequence



igure 10.2.2

- The terms in the sequence $\{n+1\}$ increase without bound.
- The terms in the sequence $\{(-1)^{n+1}\}$ oscillate between -1 and 1.
- The terms in the sequence $\{n/(n+1)\}$ increase toward a "limiting value" of 1.
- The terms in the sequence $\left\{1+\left(-\frac{1}{2}\right)^n\right\}$ also tend toward a "limiting value" of 1, but do so in an oscillatory fashion.

10.2.2 DEFINITION. A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.

Example 2

Is the sequence $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ Converge or diverge?

Example 3

Show that the sequence $\left\{\sqrt[n]{n}\right\}_{n=1}^{\infty}$ is converge to 1

10.2.3 THEOREM. *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then*

(a) $\lim_{n\to+\infty} c = c$

(b) $\lim_{n\to+\infty} ca_n = c\lim_{n\to+\infty} a_n = cL_1$

(c) $\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

(d) $\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(e) $\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(f) $\lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2}$ (if $L_2 \neq 0$)

Example 4

Determine whether the sequence converge or diverge.
If it converge find the limit.

$$a)\left\{1-\frac{n^2}{4}\right\}_{n=1}^{\infty} \qquad b)\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$$

10.2.4 THEOREM. *A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*

Example 5

Show that the sequence $\left\{(-1)^{n+1}\dfrac{n}{2n+1}\right\}_{n=1}^{\infty}$ is diverge

HW Show that the sequence $1, \dfrac{1}{2}, 1, \dfrac{1}{3}, 1, \dfrac{1}{4}, 1, \ldots$ is diverge

10.2.5 THEOREM (*The Squeezing Theorem for Sequences*). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \le b_n \le c_n \quad (\text{for all values of } n \text{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

Example 6

Show that the sequence $\left\{ \frac{\cos^2 n}{3^n} \right\}_{n=1}^{\infty}$ is converge

HW See example 8 p631

10.2.6 THEOREM. *If* $\lim_{n \to +\infty} |a_n| = 0$, *then* $\lim_{n \to +\infty} a_n = 0$.

Sequences Defined Recursively

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n} \right)$$