

## 10.9 Binomial Series

If  $m$  is a real number, then the Maclaurin series for  $(1 + x)^m$  is called the *binomial series*; it is given by

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots$$

In the case where  $m$  is a nonnegative integer, the function  $f(x) = (1 + x)^m$  is a polynomial of degree  $m$ , so

$$f^{(m+1)}(0) = f^{(m+2)}(0) = f^{(m+3)}(0) = \cdots = 0$$

and the binomial series reduces to the familiar binomial expansion

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + x^m$$

which is valid for  $-\infty < x < +\infty$ .

It can be proved that if  $m$  is not a nonnegative integer, then the binomial series converges to  $(1 + x)^m$  if  $|x| < 1$ . Thus, for such values of  $x$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-k+1)}{k!}x^k + \dots$$

or in sigma notation,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\dots(m-k+1)}{k!}x^k \quad \text{if } |x| < 1$$

**Example 1** Find binomial series for

a)  $\frac{1}{(1+x)^2}$

b)  $\frac{1}{\sqrt{1+x}}$

**Example 2**

a) Show that the power series of  $f(x) = -\ln(1-x)$  is  $\sum_{k=1}^{\infty} \frac{x^k}{k}$

With interval of convergence  $(-1, 1)$ .

b) Using part (a) find the sum of the series :  $\sum_{k=1}^{\infty} \frac{1}{k5^k}$

MACLAURIN SERIES	INTERVAL OF CONVERGENCE
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$	$-1 < x < 1$
$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \dots$	$-1 < x < 1$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$	$-\infty < x < +\infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$-\infty < x < +\infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$-\infty < x < +\infty$
$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$-1 < x \leq 1$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$-1 \leq x \leq 1$
$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$	$-\infty < x < +\infty$
$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$	$-\infty < x < +\infty$
$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!} x^k$	$-1 < x < 1^*$ $(m \neq 0, 1, 2, \dots)$