10.8 Maclaurin and Taylor Series; Power Series

DEFINITION. If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots \quad (1)$$

the ***Taylor series for f about x = x₀***. In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots \quad (2)$$

in which case we call it the ***Maclaurin series for f***.

For example Maclaurin series for *sinx is* $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$

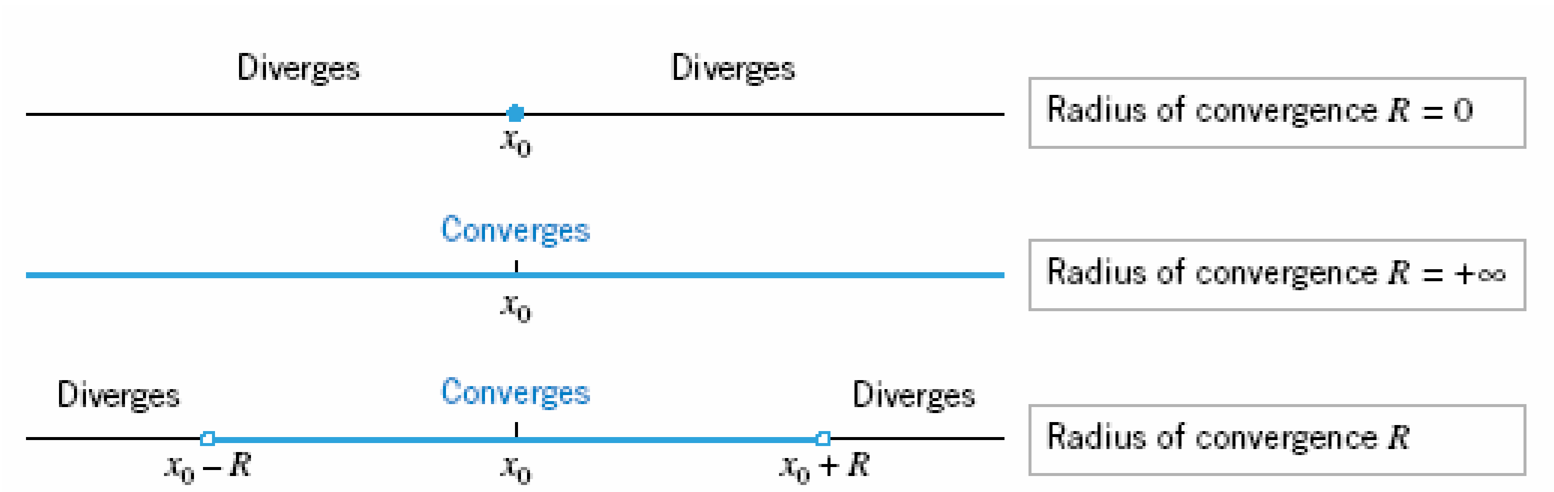
Power Series in $x - x_0$

THEOREM. *For a power series $\sum c_k(x - x_0)^k$, exactly one of the following statements is true:*

(a) *The series converges only for $x = x_0$.*

(b) *The series converges absolutely (and hence converges) for all real values of x.*

(c) *The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*



Example 1

Find the interval of convergence and radius of convergence of the following power series.

$$a)\sum_{k=0}^{\infty}\frac{x^k}{5^k} \qquad b)\sum_{k=0}^{\infty}\frac{x^k}{k!} \qquad c)\sum_{k=1}^{\infty}\frac{x^k}{\sqrt{k}}$$

$$d)\sum_{k=0}^{\infty}(-1)^k\frac{(x-3)^k}{k+1} \qquad e)\sum_{k=0}^{\infty}(-1)^k\frac{k^k(x-3)^k}{k+1}$$

Bessel Functions

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$$

$$J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2^{2k+1}(k!)(k+1)!}$$