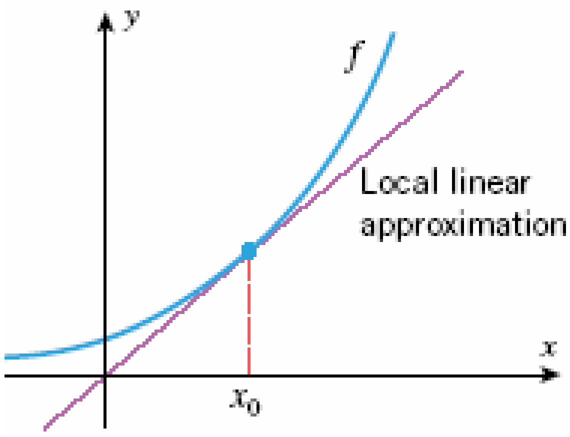
10.7 Maclaurin and Taylor Polynomials



$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$p(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0) = p(x_0), \text{ and } f'(x_0) = p'(x_0)$$

Local Quadratic Approximations of f at $x = x_0$

$$f(x) \approx p(x) = c_0 + c_1 x + c_2 x^2, \text{at } x = x_0$$

$$\text{satisfying } f(x_0) = p(x_0),\ f'(x_0) = p'(x_0), \text{and } f''(x_0) = p''(x_0)$$

For simplicity, take $x_0 = 0$

$$p(x) = c_0 + c_1 x + c_2 x^2 \Rightarrow p(0) = c_0 = f(0)$$

$$p'(x) = c_1 + 2c_2 x \Rightarrow p'(0) = c_1 = f'(0)$$

$$p''(x) = 2c_2 \Rightarrow p''(0) = 2c_2 = f''(0)$$

$$f(x) \approx p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

10.1.1 PROBLEM. Given a function f that can be differentiated n times at $x = x_0$, find a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at x_0.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n \tag{6}$$

such that

$$f(0) = p(0), \quad f'(0) = p'(0), \quad f''(0) = p''(0), \ldots, \quad f^{(n)}(0) = p^{(n)}(0) \tag{7}$$

But

$$\begin{aligned}
p(x) &= c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n \\
p'(x) &= c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} \\
p''(x) &= 2c_2 + 3 \cdot 2c_3x + \cdots + n(n-1)c_nx^{n-2} \\
p'''(x) &= 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_nx^{n-3} \\
&\vdots \\
p^{(n)}(x) &= n(n-1)(n-2)\cdots(1)c_n
\end{aligned}$$

Thus, to satisfy (7) we must have*

$$
\begin{aligned}
f(0) &= p(0) &&= c_0 \\
f'(0) &= p'(0) &&= c_1 \\
f''(0) &= p''(0) &&= 2c_2 = 2!c_2 \\
f'''(0) &= p'''(0) &&= 3 \cdot 2c_3 = 3!c_3 \\
&\vdots \\
f^{(n)}(0) &= p^{(n)}(0) &&= n(n-1)(n-2)\cdots(1)c_n = n!c_n
\end{aligned}
$$

which yields the following values for the coefficients of $p(x)$:

$$c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2!}, \quad c_3 = \frac{f'''(0)}{3!}, \ldots, \quad c_n = \frac{f^{(n)}(0)}{n!}$$

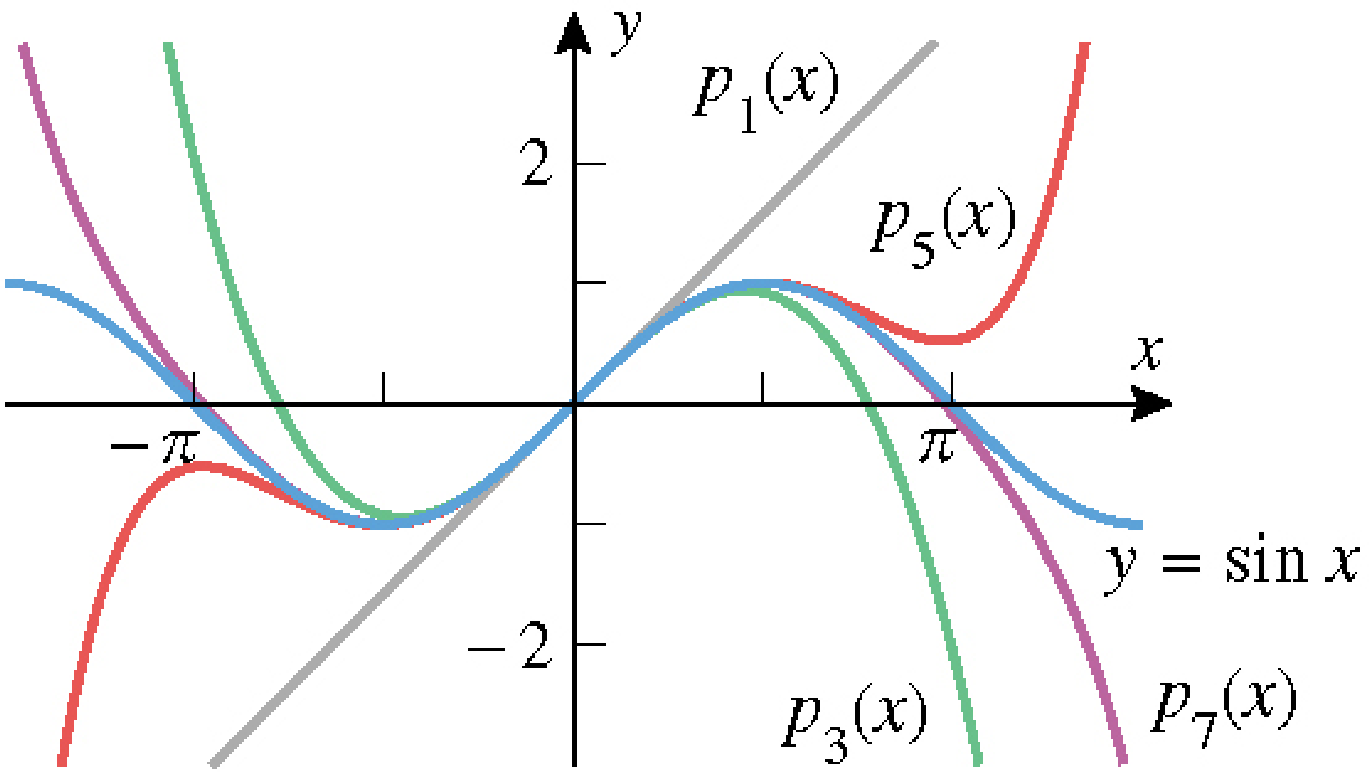
DEFINITION. If f can be differentiated n times at 0, then we define the ***nth Maclaurin polynomial for f*** to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{8}$$

This polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.

$$p(x) = \sum_{k=0}^{n} \frac{f^{k}(0)}{k!}x^k$$

Example 1 Find the Maclaurin polynomial for $f(x)=\sin x$



Example 2 Find the Maclaurin polynomial for $f(x)=\dfrac{1}{1-x}$

DEFINITION. If f can be differentiated n times at x_0, then we define the ***nth Taylor polynomial for f about*** $x = x_0$ to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$p(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!}\left(x - x_0\right)^k$$

Example 3 Find the Taylor polynomial for $f(x) = \ln x,\ at\ x_0 = 1$

HW Find the Maclaurin polynomial for $f(x) = \cosh x$