10.6 Alternating Series; Conditional Convergence

In general, an alternating series has one of the following two forms:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots \tag{2}$$

where the a_k's are assumed to be positive in both cases.

The following theorem is the key result on convergence of alternating series.

THEOREM (*Alternating Series Test*). *An alternating series of either form (1) or form (2) converges if the following two conditions are satisfied:*

(a) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots$

(b) $\lim_{k \to +\infty} a_k = 0$

Example 1

Use the alternating series test to determine whether the following series converge or diverge

$$a)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \qquad b)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{4k^2-3} \qquad c)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{4k-3}$$

Approximating Sums of Alternating Series

THEOREM. *If an alternating series satisfies the hypotheses of the alternating series test, and if S is the sum of the series, then:*

(a) *S lies between any two successive partial sums; that is, either*

$$s_n < S < s_{n+1} \quad \text{or} \quad s_{n+1} < S < s_n$$

depending on which partial sum is larger.

(b) *If S is approximated by s_n, then the absolute error $|S - s_n|$ satisfies*

$$|S - s_n| < a_{n+1}$$

Moreover, the sign of the error $S - s_n$ is the same as that of the coefficient of a_{n+1}.

Example 2

Find a value of *n* for which the *n*th partial sum is approximate the sum with error *0.00001*

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)!}$$

Absolute Convergence

DEFINITION. A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

is said to ***converge absolutely*** if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges and is said to ***diverge absolutely*** if the series of absolute values diverges.

Example 3

Determine whether the following series converge absolutely.

$$a)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \qquad b)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}$$

THEOREM. *If the series*

$$\sum_{k=1}^{\infty}|u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges, then so does the series

$$\sum_{k=1}^{\infty}u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

Example 4

$$a)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2} \qquad b)\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{3^n}$$

Conditional Convergence

THEOREM (*Ratio Test for Absolute Convergence*). *Let $\sum u_k$ be a series with nonzero terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|}$$

(a) *If $\rho < 1$, then the series $\sum u_k$ converges absolutely and therefore converges.*

(b) *If $\rho > 1$ or if $\rho = +\infty$, then the series $\sum u_k$ diverges.*

(c) *If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.*

Example 5

Use the ratio test for absolute converge to determine whether the following series converge.

$$a) \sum_{n=1}^{\infty} (-1)^n \frac{n^2+4}{3^n} \qquad b) \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{3^n}$$