10.5 The Comparison, Ratio, and Root Tests

THEOREM (The Comparison Test). *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that*

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3, \ldots, a_k \le b_k, \ldots$$

(a) *If the "bigger series" Σb_k converges, then the "smaller series" Σa_k also converges.*

(b) *If the "smaller series" Σa_k diverges, then the "bigger series" Σb_k also diverges.*

- Guess at whether the series $\sum u_k$ converges or diverges.
- Find a series that proves the guess to be correct. That is, if the guess is divergence, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if the guess is convergence, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

INFORMAL PRINCIPLE. *Constant summands in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.*

INFORMAL PRINCIPLE. *If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.*

Example 1

Use the comparison test to determine whether the following series converge or diverge.

$$a)\sum_{k=1}^{\infty}\frac{1}{2+5^k} \qquad b)\sum_{n=3}^{\infty}\frac{3}{\sqrt{n}-2} \qquad c)\sum_{n=1}^{\infty}\frac{n}{3n^3+n+1}$$

> **THEOREM** (*The Limit Comparison Test*). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*
>
> $$\rho = \lim_{k\to+\infty}\frac{a_k}{b_k}$$
>
> *If ρ is finite and $\rho > 0$, then the series both converge or both diverge.*

Example 2

Use the limit comparison test to determine whether the following series converge or diverge.

$$a)\sum_{n=10}^{\infty}\frac{1}{\sqrt{n}+3} \qquad b)\sum_{n=10}^{\infty}\frac{3n^2+5n}{\left(4n^2+2\right)2^n} \qquad c)\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^2+2k+5}}$$

THEOREM (*The Ratio Test*). *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

(a) *If $\rho < 1$, the series converges.*

(b) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(c) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 3

Use the ratio test to determine whether the following series converge or diverge.

$$a)\sum_{k=1}^{\infty} \frac{3^k}{k!} \qquad b)\sum_{k=1}^{\infty} \frac{k^k}{k!} \qquad c)\sum_{k=1}^{\infty} \frac{1}{k^2+1}$$

THEOREM (*The Root Test*). *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} (u_k)^{1/k}$$

(a) *If $\rho < 1$, the series converges.*

(b) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(c) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 4

Determine whether the following series converge or diverge.

$$a) \sum_{k=1}^{\infty} \frac{2^{2k+1}}{k^k} \qquad b) \sum_{k=1}^{\infty} \left(\frac{2k+1}{k-2} \right)^k$$

$$c) \sum_{k=1}^{\infty} \frac{\sqrt{k} \ln k}{k^3+1} \qquad d) \sum_{k=1}^{\infty} \frac{(k!)^2 2^k}{(2k+2)!}$$