10.4 Convergence Tests

THEOREM (***The Divergence Test***).

(a) *If* $\lim_{k\to+\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*

(b) *If* $\lim_{k\to+\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

THEOREM. *If the series* $\sum u_k$ *converges, then* $\lim_{k\to+\infty} u_k = 0$.

Example 1

$a) \sum_{n=1}^{\infty} \frac{n}{n+1}$

$b) \sum_{n=1}^{\infty} \frac{1}{n^2}$

$c) \sum_{n=1}^{\infty} \frac{1}{n}$

$d) \sum_{n=1}^{\infty} \frac{e^n}{n}$

THEOREM.

(a) *If* $\sum u_k$ *and* $\sum v_k$ *are convergent series, then* $\sum(u_k + v_k)$ *and* $\sum(u_k - v_k)$ *are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) *If* c *is a nonzero constant, then the series* $\sum u_k$ *and* $\sum cu_k$ *both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer* K, *the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

Example 2 Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{1}{2^{n-1}}\right)$$

Example 3

Determine whether the following series converge or diverge.

$a)\ \sum_{n=1}^{\infty}\frac{1}{n}$ $\qquad$ $b)\ \sum_{n=12}^{\infty}\frac{4}{n}$

THEOREM (The Integral Test). *Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty)$, then*

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or both diverge.

Example 4

Determine whether the following series converge or diverge.

$$a)\ \sum_{n=1}^{\infty} \frac{1}{n} \qquad\qquad b)\ \sum_{n=1}^{\infty} ne^{-n^2}$$

THEOREM (*Convergence of p-Series*).

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

Proof.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{k\to\infty} \int_1^k x^{-p}\,dx = \lim_{k\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_1^k = \lim_{k\to\infty}\left[\frac{k^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

Example 5

Determine whether the following series converge or diverge.

$$a)\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \qquad b)\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$$c)\sum_{n=1}^{\infty}\left[ke^{-k^2}+\frac{1}{k\ln x}\right]$$